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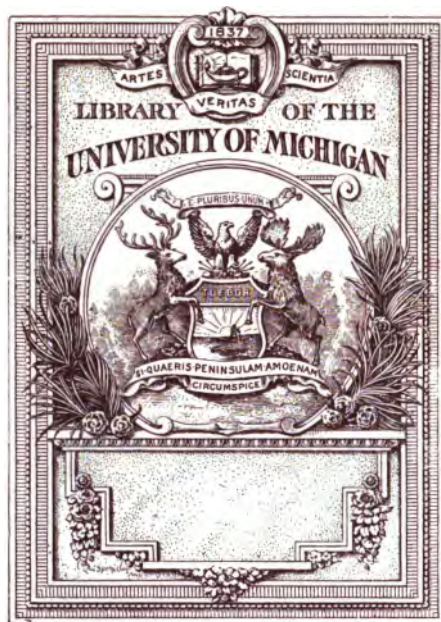
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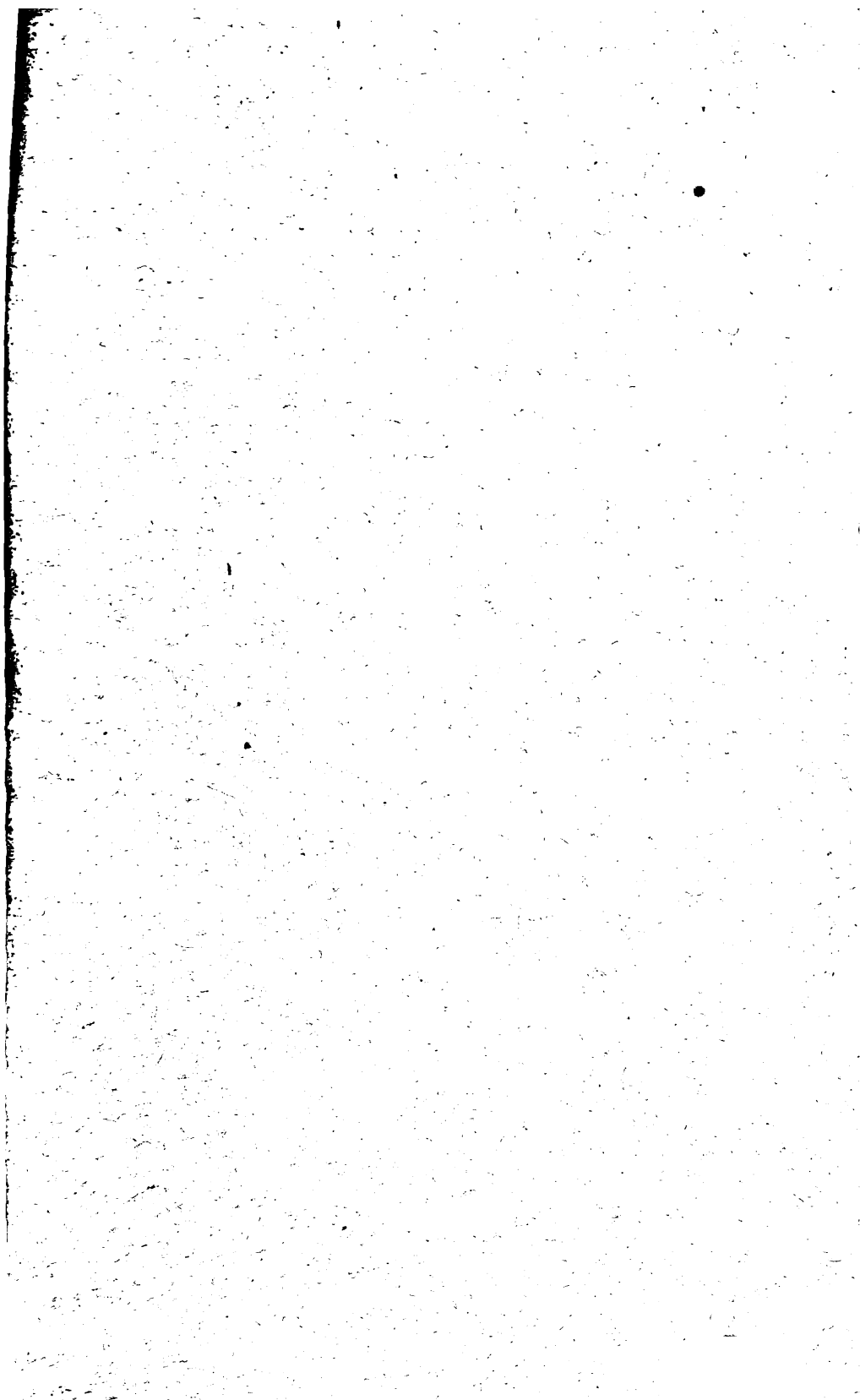


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DYNAMICS.
OR A
TREATISE ON MOTION;
TO WHICH IS ADDED
A SHORT TREATISE
ON
ATTRACTIONS.

By SAMUEL EARNSHAW, M.A.
OF ST JOHN'S COLLEGE, CAMBRIDGE.

“Dynamics, or the Science of Force and Motion, is placed at the Head
of all the Sciences.”

SIR J. F. W. HERSCHEL.

THIRD EDITION, REVISED.

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PREFACE TO THE THIRD EDITION.

For this Edition the work has been thoroughly revised, and a considerable portion of it, with the view of rendering the explanation of principles more copious, re-written. The general plan, which will be sufficiently understood from the Table of Contents, does not differ from that of the two former Editions. The Diagrams, instead of being placed as before at the end of the book, are here inserted in the text; a change which it is hoped the reader will find advantageous. The Tenth Chapter contains a Collection of Problems. These the Author has solved as much as possible by general methods for the purpose of illustrating the general principles previously laid down: for which reason he has not in any case been anxious to solve a problem by the shortest possible method, or by elegant devices adapted only to that particular case. In this Edition, several new Articles have been introduced, and amongst them a new method of considering the Perturbation of a Pendulum (Art. 157), which, as it appears more simple than any before published, is substituted in the room of the method before given. The investigations in the former editions, respecting centrifugal force, being incomplete in consequence of not determining its direction as well as its magnitude, have in this edition been replaced by other methods, which, though not so brief, are free from the objection alluded to.

The author has not uniformly adhered to one system of Differential notation, believing that each has the advantage in particular cases; but in general the subscript notation ($d_x u$ for $\frac{du}{dx}$) has been preferred, partly because without being less expressive it occupies less space in type than its rival, and is more quickly written; but chiefly because it ensures that the unpractised student in using it cannot lose sight of the independent variable, an event which is not of very uncommon occurrence when the notation $\frac{du}{dx}$ is em-

Ms. A. 9. 2. 10 v. 37

ployed. When several variables occur in an expression, it may happen to be necessary to perform the operation of differentiation upon it on several distinct hypotheses; and hence arises the necessity as well as the difficulty of inventing a system of differential notation, which shall simply but sufficiently mark the various hypotheses on which the operation is to be effected. If for example u denote an expression containing two quantities x and y , a necessity may occur of differentiating u on *four* distinct suppositions; viz. 1st, that x only is variable; 2nd, that y only is variable; 3rd, that both x and y are variable, but one of them (y) takes its variability from the other (x); and 4th, that both x and y are variable, but neither of them takes its variability from the other. These cases are of very common occurrence, and it is therefore very necessary to have a system of notation that shall distinctly express them, and at the same time be well adapted for practice. The following will answer the purpose. Let d_x be the symbol of differentiation with respect to x only, i. e. all other quantities than x being considered constant; let $\frac{d}{dx}$ be the symbol of differentiation on the supposition that all the quantities are variable so far as they can take variability from x ; and let d be the symbol of differentiation on the supposition that all the quantities are variable, without restricting them to take variability from each other; then the four cases above specified would be thus represented;

1st. $d_x u$; 2nd. $d_y u$; 3rd. $\frac{du}{dx} (= d_x u + d_y u d_x y)$; and 4th.

$$du (= d_x u dx + d_y u dy).$$

This system provides for the cases which most commonly present themselves; but others are sometimes met with, especially in Hydrodynamics and in the Planetary Theory, which it seems almost impossible to include in any general system, and they are therefore best provided for by special notation to be proposed as they occur.

CONTENTS.

CHAPTER I.

ARTS.		PAGE
1—25.	Definitions ; Laws of Motion ; Composition of Velocities .	1

CHAPTER II.

26—55.	THE IMPACT OF BODIES.	
26—37.	Laws of Impact and Elasticity...	15
38—43.	Impact of Two Bodies	21
44—47.	Impact on Fixed Planes.....	25
48.	Oblique Impact of Two Spheres.....	29
49—55.	Motion of the Centre of Gravity of a System of Bodies.....	30

CHAPTER III.

56—82.	UNIFORM ACCELERATING AND RETARDING FORCES.	
56—63.	Free Rectilinear Motion of a Particle.....	37
64—68.	Motion of a Particle on an Inclined Plane.....	41
69—75.	Motion of a Particle on a Curve Line.....	46
73—75.	Oscillation of a Particle	49
76—82.	Projectiles in Vacuo.....	54
	Problems	61

CHAPTER IV.

83—146.	VARIABLE FORCES ACTING ON A FREE PARTICLE.	
83—91.	Free Rectilinear Motion in a Plane	74
92.	Free Curvilinear Motion, referred to fixed Co-ordinates	80
93—101.	_____ revolving _____ ...	81
102—105.	_____ Tangent and Normal.	85
106.	Free Curvilinear Motion, by Parallel Forces	88
107—142.	_____ Central Forces.....	90
143—146.	Free Motion in Three Dimensions	120

CHAPTER V.

ARTS.		PAGE
147—173	CONSTRAINED MOTION OF A PARTICLE.	
147—164	Motion on a Plane Curve.....	126
152—162	Perturbation of Motion on a Cycloid.....	131
161—164	Tautochronous Curve; Curve of Equal Pressure.....	139
165	Motion on a Curve of Double Curvature.....	145
166—173	————— Curve Surface.....	147

CHAPTER VI.

174—	MOTION OF A SYSTEM OF PARTICLES.	
175, 176	Two Bodies on an Inclined Plane.....	160
177—182	D'Alembert's Principle	162
183—194	A SYSTEM OF FREE PARTICLES.....	165
184	Conservation of Motion of Centre of Gravity	166
185	————— of Areas.....	166
186	Invariable Plane.....	167
194	$\Sigma \{m(Xdx + Ydy + Zdz)\}$ an exact Differential.....	172
195—199	A RIGID SYSTEM OF PARTICLES, OR A RIGID BODY.....	173
196	Conservation of Motion of Translation	173
197	————— Rotation.....	174
200—220	A MIXED SYSTEM OF PARTICLES.....	176
200	General Equation of Motion	176
202—213	Conservation of Vis Viva	177
214—217	Principle of Least Action	183
218	————— Least Restraint.....	186
219, 220	Impulsive Action	187

CHAPTER VII.

221—250	MOMENT OF INERTIA.	
222—236	Examples	190
237—250	General Properties ..	202

CHAPTER VIII.

251—273	MOTION OF A RIGID BODY ABOUT A FIXED AXIS.	
252—273	Oscillation of a Compound Pendulum	211
252	————— Forces not impulsive.....	211
253	————— Forces impulsive.....	212
254—259	Pressure on the Axis, Forces not impulsive.....	213
260—264	————— Forces impulsive	219
258	Permanent Axes of Rotation.....	218
263	Centre of Percussion	221
267—273	Centre of Oscillation, Kater's Pendulum	224

CONTENTS.

vii

CHAPTER IX.

ARTS.	PAGE
274—292	MOTION OF A RIGID BODY ABOUT A FIXED POINT. 232

CHAPTER X.

PROBS.	PROBLEMS.	
1— 9	Motion of a Single Particle	250
10, 11	———— two unconnected Particles	260
12, 13, 15, 17	———— a Rigid Body	263
14, 16	———— two connected Bodies	268
18	Titubation	276
19	Motion of Three Bodies	278
20	Sphere on Revolving Plane with Friction	280
21	Two Rigid Bodies, with Friction	283
22	Rigid Body with Fixed Axis	285
23	Sudden Tension	287
24	Axis of Spontaneous Rotation	288
25	Impulsive Action on a Rigid Body	290
26	Motion of a Hoop with Friction	291

CHAPTER XI.

ARTS.	MOTION IN A RESISTING MEDIUM.	
293—324		
295—296	Rectilinear Motion, Resistance $\propto$ (Vel.)	297
297—301	Projectiles, Resistance $\propto$ (Vel.)	298
302—303	Rectilinear Motion, Resistance $\propto$ (Vel.) ²	302
304—307	Projectiles, Resistance $\propto$ (Vel.) ²	304
308—311	Force in Parallel Lines	308
312—315	Central Force	310
316—322	Oscillation	313
323, 324	Perturbation of an Orbit by Resisting Medium	319

APPENDIX.

1	Universal Gravitation	323
2 — 32	Attraction of a Line, Circular Arc, Sphere, &c.	324
33 — 40	$d_f^2 V + d_g^2 V + d_h^2 V = 0$, or $-4\pi\rho$	344
41 — 56	Attraction of an Ellipsoid	349
	Miscellaneous Problems	356

ERRATA.

Page 25, line 6; *for it is read its.*

84, — 10; *for $2rG$ read $\frac{1}{2}rG$.*

85, — 20; *for S read s .*

86, — 16; *for $\{(d_1x)^2 + (d_1^2y)^2\}^{\frac{1}{2}}$ read $\{(d_1x)^2 + (d_1y)^2\}^{\frac{1}{2}}$.*

90, — 8; *for place read plane.*

91, — 10; *for $\therefore$ read $\therefore$.*

166, — 27; *for that read a fixed.*

167, — 10; *for $\Sigma(mA_s)$ read $\Sigma(mA_s)$.*

174, — 20; *for $\bar{s}$ read $\bar{s}$.*

177, — 1; *for effecting read affecting.*

196, — 3; *for element read inertia.*

— 19; *for Art. 224 read Art. 223.*

223, — 7, 22; *for vis inertia read vis inertiae.*

259, — 3; *for μ read μ' .*

By the same Author.

A TREATISE ON STATICS. Second Edition.

DYNAMICS.

CHAPTER I.

DEFINITIONS AND FIRST PRINCIPLES.

1. **DYNAMICS** is that part of **MECHANICS** which treats of the motion of material bodies.

When a particle of matter is continually changing its place in space, it is said to be *in motion*: and the line which passes through all the points of space, through which the particle has successively passed during its motion, is called its *path*.

The *place* of a particle at any time is supposed to be defined by its co-ordinates, according to the principles of Co-ordinate Geometry; and the relations or equations between these co-ordinates define its *path*.

2. **FORCE**. Whatever causes or changes motion is denominated dynamical force.

Forces are divided into several kinds; the division being founded upon some peculiarity in the circumstances under which they act. For example; Dynamical *Pressure* is the force which a body in motion exerts on an obstacle, or other body, with which it is in contact.

Impact is a sudden pressure, exerted for a very short though finite time, by a body striking against any obstacle which impedes its motion.

Attraction is the name given to the force which a body is supposed to exert upon other bodies, which are observed to move towards it whenever they are set free, although there are no discoverable connecting parts or means of action between it and them.

3. A body is said to be *free*, when there is no impediment to its moving in any direction whatever. When impedi-

ments exist so as to restrict the body from moving except in certain paths, the body is said to be in *constrained* motion.

4. The motion of a particle is either uniform or variable. It is *uniform* when equal paths are described in equal times. In other cases it is said to be *variable*.

5. *Velocity* is the *rate* at which a body performs its motion. When the motion is *uniform* its rate is constant, and from popular notions we learn that in this case the *rate* is found by dividing the number which expresses the length of any portion of the path, by the number which expresses the time of describing that portion.

Now, the number by which we express the length of a proposed line will depend upon the unit (a league, a mile, a yard, a foot, an inch, &c.) of length which we employ: and that by which we express the duration of time will depend upon the unit (a year, a day, an hour, a minute, a second, &c.) of duration of time which we employ. To prevent confusion, it is usual to employ (unless otherwise stated), a *foot* as the unit of length, and a *second* as the unit of time.

Consequently in uniform motion

$$\begin{aligned}\text{velocity} &= \frac{\text{space described, expressed in feet}}{\text{time of describing it, expressed in seconds}}, \\ &= \text{number of feet described in one second.}\end{aligned}$$

6. When the motion of a particle is variable, the rate undergoes a continual change, and cannot be found by dividing the space by the time; yet at a given point of the path, the rate would evidently be found approximately by dividing a *very small* space measured from that point by the time of describing it; and a little reflection will soon satisfy the reader that the smaller that space be taken, the more nearly will the quotient represent the rate of motion at that point.

Consequently in variable motion

$$\begin{aligned}\text{velocity} &= \frac{\text{indefinitely small space described}}{\text{time of describing it}}, \\ &= \text{limit of } \frac{\text{space}}{\text{time}}.\end{aligned}$$

7. Motion is said to be *accelerated* when the velocity continually increases as the body passes from point to point of its path. When the velocity *decreases*, the motion is said to be *retarded*.

THE LAWS OF MOTION,

The most inattentive observer of nature cannot but have perceived, that, in *many* familiar instances, the motion of bodies takes place not uncertainly, but according to rules, which, however unknown to him, are evidently fixed and certain. The amusement of shooting an arrow or throwing a stone at a distant mark is evidence of the presumption of *fixed* laws of motion. Though this presumption of the unchangeableness of the laws which regulate motion is by many persons perhaps extended only to simple and common cases of motion, Philosophers have shewn, by the aid of many experiments, that every motion which matter can assume or impart takes place in accordance with three invariable Rules, called the Laws of Motion. These we shall proceed to lay before the reader, together with such illustrations of them as may tend to convince him of their truth and generality. We must however premise a few remarks on the nature of dynamical force. Referring to Art. 2 it will be seen that it is the *cause* of motion. Now as force is not *per se* the object of any of our senses, our judgment and estimation of its nature are formed solely by reasoning upon the sensible effects which it produces. These are *motion*, *change of motion* and *pressure*. The last is the characteristic of force treated of in STATICS; but in Dynamics force is considered as producing motion or change of motion.

THE FIRST LAW OF MOTION.

8. *If a body in motion be not acted upon by any EXTERNAL force, it will move in a straight line with uniform velocity.*

This law is proved by shewing that every change in a body's motion, whether in velocity or direction, is due to the action of *external* forces. But there being no part of

space, free from the influence of external forces, it is impossible to verify this law by direct experiments, for want of a place wherein to make them. All, therefore, that we can accomplish is to shew, by changing external circumstances, that an alteration is thereby made in the change which the motion suffers; and, that by removing successively some external forces, the existence of which is thus made known, the motion becomes more nearly uniform and rectilinear. Thus, if a body be thrown along a rough pavement, the motion will deviate much from the straight line in which it is projected, and will soon cease. If the body be then thrown along a level bowling green, its motion will be more rectilinear than before, and will last a longer time. If thrown along a sheet of ice, the motion of the body is much prolonged and much more nearly uniform than in the preceding experiment. In these instances, we see, that as we have successively removed *external* hinderances, the motion has been thereby made to approach towards a verification of the law of motion we are now considering: and as even in the last experiment friction is not quite removed, and the air's resistance wholly remains, we presume that these *external* causes are sufficient to account for the want of perfect rectilinearity and uniformity in the motion: and hence we infer, that when *all* external forces are removed the motion is rectilinear and uniform.

9. From this Law it follows, that there is in matter no power to move itself or to change its motion, independently of external things: and, consequently, every change in a body's motion, whether in velocity or direction, is to be attributed to the action of *external* force.

10. If the velocity of a body's motion continually increase, it is said to be acted on by an *accelerating force*; if the velocity decrease, it is said to be acted on by a *retarding* force; if the velocity be abruptly or suddenly changed, the force which produces the change is styled *impulsive*.

11. When the motion of a body is such that the increase or decrease of velocity in equal times is constant, the force is called a *uniform* accelerating or retarding force.

THE SECOND LAW OF MOTION.

12. *If a particle of matter already in motion be acted on by any external force, the change of motion is in the direction of the force, and is in magnitude the same as if the particle had no previous motion.*

This law affirms, that the motion, communicated by an external force to a particle upon which it acts, is in no way influenced by any motion which the particle may happen to have already. Its truth is witnessed in such experiments as the following.

A pendulum is found to vibrate in the same time from east to west, north to south, or in any other direction ; and a body will fall down an inclined plane in the same time towards what point soever of the compass the face of the plane be turned. The inference from these experiments is, that the motion, which the bodies possess in common with the earth, by being carried along with it in its orbit and round its axis, produces no effect upon the oscillation of the pendulum in one case, and the descent down the plane in the other case, both of which are due to the force of gravity, the external force acting on the bodies.

Again, a body dropped from the top of a ship's mast will fall at the foot of it, and the descent will be accomplished in the same time, and, apparently to a spectator in the ship, in the same manner when the vessel is sailing along uniformly as when it is at anchor ; and, water which runs through a hole in the deck will drop upon the same point of the cabin floor in both cases ; also, a person throwing a ball from one end of the ship to the other, must throw it at the same elevation and with the same velocity in both cases. This class of experiments shews, that bodies, when loosed from immediate contact with the parts of the vessel, retain the horizontal motion which they had while in connection with it ; and that the effects of gravity and projectile force upon them are the same whether they have this horizontal motion or not.

The law holds of impulsive forces also. For, if a ball lying on the deck of the ship be struck by another, the effect as witnessed by a person on board, will be just the same in

whatever direction the blow be communicated, whether towards the head, stern, or side ; or whether the vessel be at rest or in full sail.

All the experiments agree with the law,—that the motion, which a body already has, does not effect either the direction or the velocity of the motion, which a given force will communicate to it.

13. If a force always act upon a body with the same intensity or energy, the velocity generated in any time will, by this law, be independent of that previously generated ; and therefore equal velocities will be added in equal times. Hence, by referring to Art. 11, we perceive that a uniform force always acts with the same intensity.

14. Since the motion, which one force generates, in no way interferes with the motion which another force will communicate ; it follows, that if several forces act simultaneously on a particle, each will produce its effect as though the others did not act. And, hence, if two equal uniform accelerating forces act upon a particle simultaneously in the same direction, each will produce the same velocity ; and the whole velocity generated by them will be double that due to one of them. Now, two equal forces acting in the same direction form a single force double of either of them. Hence, in a given time a double force will produce a double velocity in a given body. So, a treble force will treble the velocity ; and so on. Consequently, the velocity produced in a given body in a given time is exactly proportional to the intensity of the force which produces it : and, therefore, as we have, as yet, no mathematical measure of a uniform accelerating force, we may take the velocity generated in a unit of time as its measure : that is, uniform accelerating force = velocity generated in one second.

15. Hence, accelerating force is measured by the *rate* at which velocity increases ; and in the case of a uniform accelerating force may be found by dividing the number which expresses the velocity generated in any time by the number which expresses the time. But when the force is not uniform

the rate will undergo a continual change, and cannot be found by dividing the increase of velocity by the time. Yet the rate at a given point would evidently be found approximately by dividing a *very small* increment of velocity, generated after leaving that point, by the time in which that increment is generated; and the reader will readily perceive that the smaller is the increment of velocity which is taken, the more nearly will the quotient represent the *rate* of increase at that point.

Consequently, in variable accelerating forces,

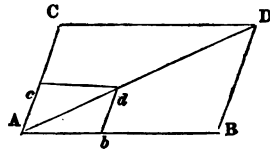
$$\begin{aligned} \text{force} &= \frac{\text{indefinitely small increment of velocity}}{\text{time of generating it}}, \\ &= \text{limit of } \frac{\text{increment of velocity}}{\text{time of generating it}}. \end{aligned}$$

Both this Art. and Art. 6 will afterwards be proved in a more formal manner. See Arts. 83, 84.

16. It will be convenient here to introduce the method of compounding two or more co-existent velocities into one resultant velocity; and of resolving a velocity into components.

COMPOSITION AND RESOLUTION OF VELOCITIES.

Let a body be placed at *A* upon the deck of a ship, which is sailing uniformly at such a rate, that by the motion of the vessel the body would in space be transferred from *A* to *B* in the time *T*, supposing it to remain at rest upon deck. Now instead of suffering the body to remain upon the deck at the point *A*, suppose we had communicated such a blow to it as would have made it move uniformly from *A* to *C* in the time *T*. Its path on deck is the same as if the ship were at rest; it is therefore a straight line (Art. 8) *AC*. Complete the parallelogram *ABDC* in space. Then by the motion of the vessel the point *C* of the deck has been conveyed to *D* in space in the time *T*; *D* is therefore the place of the body in space, at the time *T* after leaving *A*. It can be shewn also that the body describes in space the diagonal *AD* of the



parallelogram. For let Ab , Ac be the $\left(\frac{1}{n}\right)^{\text{th}}$ parts of AB , AC respectively. Then because the motions in the direction of AB , AC are uniform, and that AB , AC are described in the time T , therefore Ab , Ac are described in the time $\frac{T}{n}$, and completing the parallelogram $Abdc$, the body as before is at d at the time $\frac{T}{n}$ from leaving A .

$$\text{But } Ac : cd :: Ac : Ab$$

$$:: \frac{AC}{n} : \frac{AB}{n}$$

$$:: AC : AB$$

$$:: AC : CD.$$

Hence, joining Ad , the triangles Acd , ACD (having also the angles Acd , ACD equal) are similar, and consequently AdD is a straight line; so that at *any* time $\frac{T}{n}$ the body is situated at a point in the diagonal AD of the parallelogram whose sides AB , AC represent in magnitude and direction the two velocities which are independently communicated to it at A : hence AD represents both the direction and velocity of the motion which is the resultant of the two motions communicated at A .

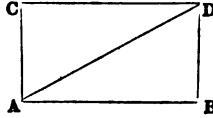
If more than two velocities are communicated at the same instant, we must find the resultant of two, and then the resultant of that and a third velocity, and so on; uniting the last found resultant with one of the remaining velocities until all have been taken into account.

17. By a *reductio ad absurdum* method, we may shew that if any straight line AD represent the velocity and direction of a body's motion, the two sides AB , BD , of any triangle constituted upon AD will represent component velocities which are equivalent to the velocity AD .

18. The most important cases of Art. 16 are those in which the original velocities act in directions at right angles to each other.

1st. Let there be two velocities AB , AC represented algebraically by a and b respectively, and let v be the resultant velocity AD .

Then, because BAC is a right angle, therefore $AD^2 = AB^2 + BD^2 = AB^2 + AC^2$;



$$\therefore v^2 = a^2 + b^2 \dots\dots\dots(1).$$

And, if the angle DAB be called α ,

$$\tan \alpha = \frac{BD}{AB} = \frac{b}{a} \dots\dots\dots(2).$$

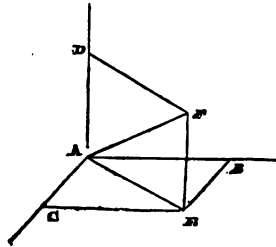
When the components a , b are given, these equations determine the magnitude and direction of their resultant velocity.

But, if v be given to find its components in two directions AB , AC inclined at angles α , and $90^\circ - \alpha$ to the direction of v , we have

$$a = AD \cdot \cos DAB = v \cos \alpha \dots\dots\dots(3),$$

$$\text{and } b = AD \cdot \sin DAB = v \sin \alpha \dots\dots\dots(4).$$

2dly. Let there be three velocities communicated simultaneously to a body at the point A in three directions AB , AC , AD at right angles to each other; and let these lines be proportional to them in magnitude. Complete the parallelogram $ABEC$, and draw the diagonal AE ; it represents the resultant of AB , AC : and completing the parallelogram $AEFD$, its diagonal AF represents the resultant of AE , AD ; that is, of AB , AC , AD . To express this analytically, let a , b , c be the velocities AB , AC , AD ; and v the component velocity AF ; and suppose α , β , γ the angles FAB , FAC , FAD which the direction of the resultant velocity makes with the directions of the component velocities respectively.



$$\begin{aligned}
 \text{Then } v^2 &= AF^2 \\
 &= AE^2 + EF^2 \\
 &= AB^2 + BE^2 + EF^2 \\
 &= a^2 + b^2 + c^2 \dots\dots\dots(5),
 \end{aligned}$$

$$\text{and } \cos \gamma = \cos FAD = \frac{AD}{AF}$$

$$\begin{aligned}
 &= \frac{c}{v} = \frac{c}{\sqrt{a^2 + b^2 + c^2}} \\
 \text{Similarly, } \cos \beta &= \frac{b}{v} = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \\
 \text{and } \cos \alpha &= \frac{a}{v} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}
 \end{aligned} \dots\dots(6).$$

When the three components a , b , c are given, (5) and (6) will give their resultant velocity both in magnitude and direction.

But when it is required to resolve a given velocity v into equivalent velocities in directions at right angles to each other, and making respectively angles α , β , γ with the direction of v , we must employ the equations

$$\begin{aligned}
 a &= v \cos \alpha \\
 b &= v \cos \beta \\
 c &= v \cos \gamma
 \end{aligned} \dots\dots(7).$$

19. We may remark, that when a line makes angles α , β , γ with three other lines which are at right angles to each other, these angles are not independent. For, by adding together the squares of the equations (6), we find

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

an equation which it is sometimes useful to employ under the forms

$$\cos^2 \beta + \cos^2 \gamma = \sin^2 \alpha,$$

$$\cos^2 \gamma + \cos^2 \alpha = \sin^2 \beta,$$

$$\cos^2 \alpha + \cos^2 \beta = \sin^2 \gamma.$$

20. From the reasoning in Art. 16, it is evident that when uniform velocities are simultaneously communicated to a body in any directions, the resultant motion is uniform and in a straight line.

21. Both the definition and the measure of accelerating force (Arts. 10, 14), are derived from the *change of a body's velocity*; they assume that the body's motion is wholly in the line in which the force acts upon it. This will not be the case if the body has been projected obliquely to the direction of the force; or if the body is constrained by any surface, string, or rod, or tension of any kind, to take an oblique course. In the former case, the velocity must be estimated in the direction in which the force acts; the other component of the velocity being that which is due to the original projectile force. In the case of constrained motion (ex. gr. a body falling down an inclined plane by the action of gravity) the change of velocity cannot be such as is exactly due to the whole force which acts upon the body. It has from this circumstance been found necessary in constrained motion to distinguish, by appropriate names, the force as *calculated* from the actual change of velocity; and the force which really *acts* upon a particle. The former is called, the *effective* accelerating force; and the latter, the *impressed* accelerating force. This last force is measured by the velocity which it *would* generate in a free particle, falling from rest, in a unit of time.

22. Accelerating force being measured by change of velocity is independent of the mass of the moving body. Yet we know from experience that the same degree of pressure applied (by the hand or otherwise) to bodies of different masses will in the same time generate in them different degrees of velocity. Accelerating force is therefore not a measure of the pressure employed, unless the masses of the bodies moved be all equal (Art. 14). The object of the next law of motion is to find by experiment what is the general relation between the three elements here mentioned; viz. pressure, mass, and accelerating force.

THE THIRD LAW OF MOTION.

23. *The pressure (measured as in Statics by the weight which it can support) which produces motion in any body is equal to the product of the mass of the body into the accelerating force.*

If equal pressures be employed, under similar circumstances, upon bodies equal in every respect, the same velocity (there being no reason to the contrary) will be produced in each in the same time: and if the bodies be adjacent at starting, they will continue to be adjacent during the entire continuance of their motion; and we may suppose them to coalesce in unequal groups, at any period of their motion, without affecting their subsequent motion. Now when the coalition has taken place, we have a series of unequal bodies, all moving with the same velocity, the mass of each of which is exactly proportional to the number of original small bodies in the group which coalesced in forming it. The whole pressure also, which acts upon each of these new bodies, is in the same proportion; that is, the pressure on any one is proportional to its mass. Consequently,

When the pressures applied to bodies are proportional to the masses to which they are applied, equal velocities will be generated in all the bodies in the same time: in other words,

The pressure required to produce a given accelerating force in bodies of different masses varies as the mass. ... (A).

Again, let two bodies, whose masses are represented by A and B , be connected by a string; and let B be laid upon a very smooth horizontal table, while A is suffered to hang down over the edge of it. If the system thus arranged be left to itself, A will begin to descend, drawing B along the table with a velocity equal to its own. Now take a portion x from A and annex it to B , and with the bodies $A - x$ and $B + x$, let the experiment be repeated: it is found that, (making due allowance for the want of perfect smoothness in the table) the velocity acquired by the system in a given time is proportional to $A - x$, the weight that hangs down, whatever be the value of x . Now from the arrangement of these experiments, the

mass moved is always the same, being equal to $A + B$: and the pressure which produces motion in this mass is the weight of that portion only which hangs down, $A - x$; for if that be supported by a stand placed under it, no motion will ensue when the system is left to itself. Hence, *if bodies of equal masses be put in motion by unequal pressures, the accelerating forces produced thereby are proportional to the pressures which respectively produce them*: in other words,

The pressures required to produce different accelerating forces in equal masses, vary as the accelerating forces.

Uniting this variation with that marked (A) we find this result,

pressure $\propto$ (mass-moved) $\times$ (accelerating force produced by it).

As we have not as yet fixed upon a measure of mass, we are at liberty to assume the unit of mass to be such as will turn the above variation into an equation:

$$\therefore \text{pressure} = (\text{mass}) \cdot (\text{accelerating force}).$$

24. The product on the right-hand side of this equation is often called the *moving force*, to distinguish it from accelerating force: and then the above law of motion is stated thus,

The pressure which produces motion varies as the moving force.

25. We have shewn in *Statics*, that every pressure can be expressed by a weight; we shall shew hereafter how the accelerating force of gravity can be found with great accuracy: these two furnish us with an expression for the mass of any body. For if W be the weight of a body, M the measure of its mass, and g the accelerating force of gravity (see Art. 64);

$$W = Mg,$$

$$\therefore M = \frac{W}{g}.$$

GENERAL REMARK ON THE LAWS OF MOTION.

When we consider that all the experiments adduced in proving the Laws of Motion are of a very rough character, and at the most are only proofs that the Laws hold good in those particular instances, some doubt may remain on the mind of the student respecting their generality. Perhaps the consideration most likely to remove any doubt of this kind is, that they have been taken as the basis of an endless variety of calculations respecting the motions of bodies under an almost infinite variety of circumstances:—they have been applied in computing the places of the Sun, Moon, and planets;—in predicting Eclipses, occultations and other celestial phænomena;—and in explaining the motion of the Earth about its own axis;—in all which cases a great number of forces, acting in different ways, and under different circumstances, are to be taken account of; which therefore leads to calculations of great length, complexity and delicacy, in which an error in first principles must needs be multiplied so often, and be so interwoven into every part, as utterly to vitiate the results;—yet in every instance have these results, thus obtained, been found to differ from the results of observation by quantities of a most minute character,—the discrepancies never being greater than what might reasonably be attributed to the errors of data and observation. The irresistible inference to be drawn from this most exact agreement between theory and experiment is, that the Laws of Motion which formed the basis of the calculations are true, and of universal application to nature.

CHAPTER II.

ON THE IMPACT OF BODIES.

26. WHEN two bodies moving directly towards each other meet, or if one overtake the other, upon their surfaces first coming in contact, the motion of each receives a check, which to our senses appears to take place instantaneously. That a finite, though very small, time is really occupied in effecting the changes in their motions, is rendered certain by a closer examination of the experiment. For, if two ivory balls, one of which is stained with ink, be made to touch each other, at the point of contact, upon the clean ball, there will be seen an extremely minute spot of ink. But if they be made to strike against each other with considerable velocities, the spot of ink will be very much enlarged; and the greater the striking velocities, the larger will be the spot. Now, as two spheres can only *touch* each other in a single point, the appearance of a large round spot transferred from the stained ball to the clean ball, shews that their figures during the impact have been compressed at and near the point of first contact; hence, during the impact the centres of the balls have approached nearer to each other than the sum of their radii, which is their distance when merely in contact. This approach through a finite space being accomplished with finite velocities, must necessarily have occupied a *finite time*. We can now see also how the change in the velocities is effected. At first meeting the balls begin to press against each other;—this pressure goes on increasing as the balls become more compressed, until the compression has reached its maximum, at which moment the pressure has also obtained its greatest value, and the approach of their centres having ceased, the velocities of the balls are equal. At this moment the natural effort of the substance of the balls to restore their former figure (that of perfect spheres) causes them to thrust one another apart with considerable violence, (yet this operation also occupies a finite

time,) and produces the phenomenon of rebounding. That property of matter by which the balls make this effort to regain their former figure is called *elasticity*. If the balls regain their figure with the same force as was required to compress them, the elasticity of the substance of which they are composed is said to be *perfect*. No substance of this kind has hitherto been found in nature. All bodies are therefore considered as *imperfectly elastic*.

27. During the whole process of compression and separation detailed above, there exists a mutual pressure, which, though continually varying, is always the same for both, and which is employed in retarding (or accelerating as the case may be) the motions of the balls until they separate.

Now the retarding force which results from this pressure

$$= \frac{\text{pressure}}{\text{mass on which it acts}} \dots\dots\dots (\text{Art. 23.})$$

Hence, the numerator of this fraction being the same for both the bodies, if their masses be A and B ,

$$\frac{\text{retarding force on } A}{\text{retarding force on } B} = \frac{B}{A}.$$

But in Art. 14 it is shewn that the velocities generated or destroyed in equal times by uniform forces are proportional to the accelerating or retarding forces which generate or destroy them. Hence however two forces may *vary* in magnitude, if their *ratio* remains constant, the indefinitely small increments or decrements of velocity which they generate or destroy in very small equal times, during which they may be supposed uniform, have the same ratio, and hence the whole velocities generated or destroyed have this ratio. Consequently

$$\frac{\text{vel. destroyed in } A \text{ during impact}}{\text{vel. destroyed in } B \text{ during impact}} = \frac{B}{A};$$

$$\therefore A \cdot (\text{vel. lost by } A) = B \cdot (\text{vel. lost by } B).$$

This is the form of the equation when the bodies *meet* at impact; but if A *overtake* B , it is thus expressed,

$$A \cdot (\text{vel. lost by } A) = B \cdot (\text{vel. gained by } B).$$

The product of mass into velocity is called *momentum*, and hence the result of this article may be thus stated in words,

In the direct impact of two bodies the momentum lost by one of them is equal to that lost or gained by the other.

28. There is no part of the investigation in the last article which requires the bodies to be in actual contact: if by any other means they are enabled to exert in the line which joins them such an action upon each other, as is equivalent to an equal pressure applied to each in opposite directions, the same result will follow, viz. that

$$A. (\text{vel. lost or gained by } A) = B. (\text{vel. lost or gained by } B).$$

29. It is easily shewn that the *attraction* which two bodies exert upon each other is of this nature; for if the attracting bodies be placed at the opposite ends of a rigid rod no motion will ensue, which proves that they exert equal pressures upon the rod in opposite directions. The rod, as long as it remains between the bodies, prevents motion, but the instant it is removed, the same forces which caused them to exert equal pressures, now cause them to rush towards each other. Hence the force of attraction is equivalent to an equal pressure applied to each body in opposite directions. Consequently when two bodies *A, B* move towards each other by mutual attraction,

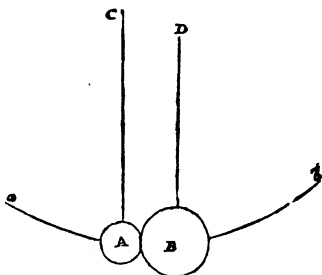
$$A. (\text{vel. gained by } A) = B. (\text{vel. gained by } B).$$

Hence the result of Art. 27 admits of a more general enunciation; thus,

In the direct mutual action of two bodies the momentum lost or gained by one is equal to that lost or gained by the other.

30. This was formerly the third Law of Motion, and was stated thus, "Action and Reaction are equal and opposite." We may regard it as a deduction from our second and third Laws, and as it is a result which admits of many simple and varied experimental means of examination, it furnishes us

with easy tests of the truth of those laws from which it has been derived. Thus, let A, B be two balls of different substances, magnitudes, and weights, suspended by two strings CA, DB from two such points C, D that when at rest the balls just touch each other and have their centres in the same horizontal line. This arrangement is necessary in order to insure *direct* impact when the balls are elevated and suffered to fall through such circular arcs aA, bB , (whose centres are C, D) that they may strike together at the moment when they reach their lowest positions. By having the strings of different lengths, if necessary, and letting the bodies start at proper times, we may insure their meeting together at the lowest points of their respective arcs, and with any velocities we please. After striking together they will rebound, and by measuring the respective altitudes through which they ascend we shall be able to calculate the velocities with which they rebounded, that is, their velocities after impact. The velocities before impact are also known, and thus we can ascertain whether the momenta lost by the balls are equal. This experiment has been repeatedly tried, and been attended with the most satisfactory results, and hence affords us strong evidence of the truth of these Laws of Motion. If it be thought necessary, the effect of compression may be examined apart from that of elasticity by fixing a steel point in one of the bodies at the point of impact, which will cause them to adhere together after striking.



31. But, again, it is evident from Art. 27, that if two bodies A, B are always subject to equal pressures, by whatsoever means those pressures are produced, if only they act in the directions of motion of the bodies to which they are respectively applied, the same result follows; and consequently, if A, B act upon each other through any machine, so as to satisfy these conditions, viz. that equal pressures are always exerted on them in their respective directions of motion, it is

true in this case also, as in the case of direct action, that *the momentum lost or gained by one of them is equal to that lost or gained by the other.*

32. If n bodies of the same mass move with equal velocities, the *quantity of motion* in each is the same: and therefore the whole quantity of motion in the system is n times that existing in one of the bodies. Hence, when a body or system of bodies moves with a given velocity,

$$\text{quantity of motion} \propto \text{mass}.$$

It is evident that if the velocity of a body be increased in any proportion, the quantity of motion is increased in the same proportion. Hence when the mass is given,

$$\text{quantity of motion} \propto \text{velocity}.$$

Uniting this variation with the last, we find that, whatever be the mass and velocity,

$$\begin{aligned} \text{quantity of motion} &\propto (\text{mass}) \cdot (\text{velocity}) \\ &\propto \text{momentum}. \end{aligned}$$

33. Now if force be the cause of motion it seems natural to expect, since cause and effect are necessarily proportional, that the *quantity of force employed* ought to be exactly proportional to the *quantity of motion produced or destroyed*. We shall shew that this follows from the measures of force and motion which we have given.

Let the whole duration of a pressure be divided into extremely short intervals $t_1, t_2, t_3, \dots$ each so small that the pressure exerted during any one of them may be considered constant or uniform. Let $P_1, P_2, P_3, \dots$ be these pressures, exerted during the 1st, 2nd, 3rd, $\dots$ intervals respectively; and let m be the mass moved.

Then the whole *quantity of pressure* exerted may be considered equal to $P_1 t_1 + P_2 t_2 + P_3 t_3 + \dots$. But if $f_1, f_2, f_3, \dots$ be the accelerating forces resulting from the pressures $P_1, P_2, P_3, \dots$ respectively, and $v_1, v_2, v_3, \dots$ be the velocities generated by them in the times $t_1, t_2, t_3, \dots$ during which they respectively act, then

$$\begin{aligned}
 P_1 t_1 + P_2 t_2 + P_3 t_3 + \dots &= m f_1 t_1 + m f_2 t_2 + m f_3 t_3 + \dots \text{ Art. 23,} \\
 &= m v_1 + m v_2 + m v_3 + \dots \text{ Art. 15,} \\
 &= m \cdot (\text{whole velocity generated or} \\
 &\quad \text{destroyed}) \\
 &= \text{momentum generated or destroyed;}
 \end{aligned}$$

$\therefore$ whole *quantity* of pressure

= momentum generated or destroyed,

$\propto$ quantity of motion generated or destroyed.

34. That which is here called the whole quantity of pressure is, in the case of impact, called the *impulse* or the *blow*; for when a body strikes against an obstacle and loses part of its motion, the blow sustained by the obstacle seems to be properly measured by the quantity of motion destroyed. Wherefore

impulse = momentum generated or destroyed.

35. Impinging bodies are supposed to be spherical, and of uniform density, and of the same elasticity unless it be otherwise mentioned.

The *relative velocity* of two bodies, is that velocity with which they *approach* towards, or *recede* from each other. It is therefore the *sum* of their absolute velocities when they move in opposite directions; and the *difference* of them when they move in the same direction.

36. It is found by experiments made, in the manner detailed in Art. 30, with bodies of the same substance, that the relative velocity of two such bodies after impact bears a constant ratio to their relative velocity before impact, whatever be the absolute velocities with which they strike together. This ratio however is not the same for all substances. Tables have been formed of its value for different substances.

It is taken as the measure of the elasticity of the substance, and in this work will be represented by λ , the value of which is supposed to be known when the nature of the substance employed is given.

Substance.	Value of λ .	Substance.	Value of λ .
Glass	·94	Bell Metal	·67
Ivory	·81	Cork.....	·65
Hardened Steel	·79	Brass	·41
Cast Iron	·73	Lead	·20

37. The *plane* of impact of two bodies is that which touches them both at the point of their mutual contact.

If the paths of their centres before impact are not *both* perpendicular to this plane they are said to impinge *obliquely* upon each other.

Bodies are said to be *inelastic* which do not separate after impact.

38. **PROB.** *Given the absolute velocities of two inelastic bodies, to find their common velocity after direct impact.*

Let A, B be their masses; a, b their respective absolute velocities before impact, and v their common velocity after impact. (All these velocities are estimated in the direction of A 's motion; so that if B 's motion should be in the opposite direction to that of A , b will be a negative quantity.)

A 's absolute velocity after impact $= v$, and therefore the velocity lost by $A = a - v$; hence the momentum lost by A

$$= A(a - v).$$

B 's absolute velocity after impact $= v$, and therefore the velocity gained by $B = v - b$; hence the momentum gained by B

$$= B(v - b).$$

But by Art. 27, these momenta are equal;

$$\therefore A(a - v) = B(v - b);$$

$$\therefore v = \frac{Aa + Bb}{A + B}.$$

REM. The impulse, or blow with which the bodies strike each other, may be found from the definition of Art. 34: for,

being measured by the momentum lost by either of them, it is equal to $A(a - v)$,

$$\text{which} = \frac{AB(a - b)}{A + B}.$$

39. **PROB.** *Given the absolute velocities of two elastic bodies before direct impact, to find the absolute velocity of each after impact.*

Let A, B be their masses; λ their elasticity; a, b their respective absolute velocities before impact; a', b' their velocities *after* impact: (all velocities estimated in direction of A 's motion).

Then the velocity lost by $A = a - a'$

and the velocity gained by $B = b' - b$;

$\therefore$ the momentum lost by $A = A(a - a')$

and the momentum gained by $B = B(b' - b)$.

Consequently by Art. 27,

$$A(a - a') = B(b' - b) \dots \dots \dots (1).$$

Now the relative velocity before impact $= a - b$, and after impact $= b' - a'$. (We write $b' - a'$, and not $a' - b'$ because A is supposed to strike against B and to drive B before it, which makes b' greater than a' .)

Hence by Art. 36,

$$\frac{b' - a'}{a - b} = \lambda \dots \dots \dots (2);$$

$$\therefore b' = a' + \lambda(a - b)$$

which substituted in (1), gives

$$a' = \frac{Aa + Bb}{A + B} - \lambda \cdot \frac{B(a - b)}{A + B}.$$

Similarly by substituting $b' - \lambda(a - b)$ for a' in (1) we find,

$$b' = \frac{Aa + Bb}{A + B} + \lambda \cdot \frac{A(a - b)}{A + B}.$$

REM. The impulse, or blow with which the bodies strike together,

$$= A(a - a') = \frac{(1 + \lambda) AB(a - b)}{A + B}.$$

40. If B had been at rest before impact, $b = 0$; and writing this value of b in last article,

$$a' = \text{the vel. of } A \text{ after impact} = \frac{(A - \lambda B)a}{A + B},$$

$$b' = \text{the vel. of } B \text{ after impact} = \frac{(1 + \lambda)Aa}{A + B}.$$

Consequently, if A and B should be so related that $A - \lambda B = 0$, or $\frac{A}{B} = \lambda$, A after striking against B will remain at rest, and B will move on with a velocity equal to λa ; which result is obtained by writing $B\lambda$ for A , in the above value of b' .

41. If B should happen to be indefinitely greater than A , $b' = 0$, that is, B is not affected by A 's impact; but since

$$a' = \frac{\left(\frac{A}{B} - \lambda\right)}{\frac{A}{B} + 1} a = -\lambda a.$$

A is reflected from B with the velocity λ times the velocity of impact.

42. PROB. *If each body be multiplied by the square of its velocity, the sum of the products so formed before impact will be equal to the sum so formed after impact, when the bodies are perfectly elastic; but the former sum will be greater than the latter if the bodies are imperfectly elastic.*

1st. Let the bodies be perfectly elastic, then $\lambda = 1$;

$$\text{and } \frac{b' - a'}{a - b} = 1 \dots \dots \dots \text{Art. 36;}$$

$$\therefore b' - a' = a - b,$$

$$\text{or } a + a' = b' + b.$$

$$\text{But } A(a - a') = B(b' - b) \dots \text{Art. 27.}$$

Multiply the last two equations together;

$$\therefore A(a^2 - a'^2) = B(b'^2 - b^2),$$

$$\text{or } Aa^2 + Bb^2 = Aa'^2 + Bb'^2,$$

which proves the proposition for this case.

2dly. Let the bodies be imperfectly elastic; then, by Art. 36,

$$\frac{b' - a'}{a - b} = \lambda = \frac{\lambda}{1};$$

$$\therefore \frac{(a + a') - (b' + b)}{(a - a') + (b' - b)} = \frac{1 - \lambda}{1 + \lambda};$$

$$\therefore (a + a') - \frac{1 - \lambda}{1 + \lambda} (a - a') = (b' + b) + \frac{1 - \lambda}{1 + \lambda} (b' - b).$$

$$\text{But } A(a - a') = B(b' - b).$$

Hence, multiplying the last two equations together,

$$A(a^2 - a'^2) - \frac{1 - \lambda}{1 + \lambda} \cdot A(a - a')^2 = B(b'^2 - b^2) + \frac{1 - \lambda}{1 + \lambda} \cdot B(b' - b)^2;$$

$$\therefore Aa^2 + Bb^2 = Aa'^2 + Bb'^2 + \frac{1 - \lambda}{1 + \lambda} \cdot \{A(a - a')^2 + B(b' - b)^2\}.$$

Consequently, $Aa^2 + Bb^2$ is greater than $Aa'^2 + Bb'^2$.

43. PROB. *There are two bodies whose masses A, C are given, C being at rest while A is moving towards it with a given velocity a; it is required to find the mass of a body, which being interposed between A and C, shall, after being struck by A, communicate the greatest possible velocity to C.*

Let B be the mass of the interposed body, b' its velocity after being struck by A; then, by Art. 40, since B is at rest before being struck by A,

$$b' = \frac{(1 + \lambda)Aa}{A + B}.$$

The same formula gives the velocity of C after being struck by B

$$= \frac{(1 + \lambda)Bb'}{B + C} = \frac{(1 + \lambda)B}{B + C} \cdot \frac{(1 + \lambda)Aa}{A + B}.$$

This quantity is to be a maximum; or, omitting the constant factors, $\frac{B}{(B + C)(A + B)}$ is to be a maximum; and therefore it is reciprocal $\frac{(B + C)(A + B)}{B}$, which

$$= A + C + B + \frac{AC}{B},$$

is to be a minimum, by the variation of B . Hence, according to the principles of the Differential Calculus, equating to zero, the differential coefficient of this expression with regard to B , we have

$$1 - \frac{AC}{B^2} = 0,$$

$$\text{or } B^2 = AC;$$

that is, the mass of the interposed body must be a mean proportional between the masses of the given bodies.

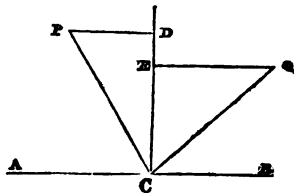
IMPACT ON FIXED PLANES.

DEF. The angle which the direction of a body's motion makes with a perpendicular upon the plane against which it impinges is called, before impact, the angle of *incidence*; and after impact, the angle of reflection.

44. PROB. *An imperfectly elastic body, moving with a given velocity in a given direction, strikes against a smooth fixed plane at a given point; to determine the velocity and direction of its motion after impact.*

Let AB be the plane; C the point of impact, CD a perpendicular to the plane at that point, PC the path of the body *before* impact; take this line to represent the given velocity of the body; draw PD parallel to the plane, and in CD

take a point E such that $\frac{CE}{CD} = \lambda$;



from E draw $EQ = PD$, and join C, Q ; CQ is the velocity and direction of the motion *after* impact.

For, the motion of the body, represented by PC , may be resolved into the two motions PD, DC (Art. 17), the effects of which (as in the second Law of Motion) we may consider separately. The motion PD being parallel to the plane will not be altered by impact, inasmuch as the plane is smooth, and cannot offer any resistance parallel to itself; EQ , which is equal to PD , will therefore represent after impact, the velocity of the body, parallel to the plane. But the velocity DC would cause the body to impinge *directly* upon the fixed plane; and therefore the velocity after impact in the direction $CD = \lambda \cdot CD^*$ (Art. 41) $= CE$; hence after impact, CE represents the velocity which is perpendicular to the plane. The velocities CE, EQ compounded into one (Art. 16), give CQ for the direction and velocity of the body after impact. To express this result analytically, let v be the velocity before impact, v' the velocity after impact; θ the angle of incidence PCD , and θ' the angle of reflection QCD : then

$$\tan \theta' = \frac{EQ}{CE} = \frac{PD}{\lambda \cdot CD} = \frac{1}{\lambda} \cdot \tan \theta \dots \dots \dots (1)$$

$$\begin{aligned} \text{and } v' \sin \theta' &= EQ \\ &= PD \\ &= v \sin \theta. \end{aligned}$$

* Art. 36 properly applies only to the *relative* motion of two bodies; yet the experimental results there given will be applicable in the present case, if we suppose AB to be the face of a body whose mass is indefinitely great in comparison of that of the impinging body; on this principle, the above equation is taken from Art. 41.

Substituting for $\sin \theta'$ from (1), we find

$$v'^2 = v^2 (\sin^2 \theta + \lambda^2 \cos^2 \theta),$$

$$\text{or } v' = v (\sin^2 \theta + \lambda^2 \cos^2 \theta)^{\frac{1}{2}} \dots \dots \dots (2).$$

Equation (1) determines θ' , and (2) gives v' .

REM. If the ball be perfectly elastic, $\lambda = 1$, and $\theta = \theta'$, $v = v'$; consequently, the angle of reflection is equal to the angle of incidence, and the velocity is unchanged by impact except in direction.

45. If the ball were inelastic, CE would vanish, and after impact it would run along CB with the

$$\text{vel. } EQ = PD = v \sin \theta :$$

in which case the velocity lost by impact would

$$= v - v \sin \theta$$

$$= v(1 - \cos PCA)$$

$$= v \cdot \text{vers } PCA.$$

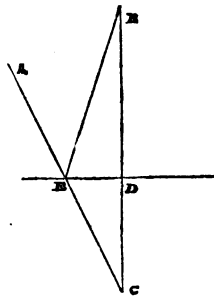
46. **PROB.** *To find the direction in which an elastic body must be projected from a given point; so that, after reflection at a given plane surface, it may pass through another given point.*

Suppose the body is to be projected from A against the plane ED , so as after reflection to pass through B . Draw BC perpendicular to ED , making $\frac{BD}{DC} = \lambda$; join AC ; it is the direction in which the body must be projected from A .

For join EB , and project the body in the direction AE ; then

$$\tan \text{ angle of reflection} = \frac{1}{\lambda} \cdot \tan \text{ angle of incidence} \dots \dots (\text{Art. 44})$$

$$= \frac{1}{\lambda} \cdot \tan ECD$$



$$= \frac{1}{\lambda} \cdot \frac{ED}{CD}$$

$$= \frac{ED}{BD}$$

$$= \tan EBD;$$

$\therefore$ angle of reflection = EBD ,

hence EB is the direction of reflection, and the body is reflected from E through B .

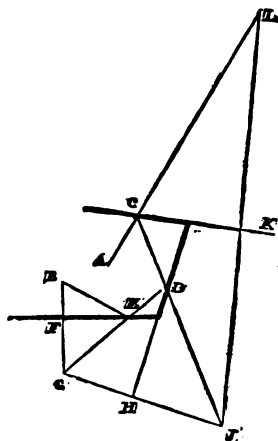
47. PROB. *To find the direction in which an elastic body must be projected from a given point; so that after reflection at any number of given plane surfaces in a given order it may pass through a given point.*

Let A be the point from which the body is to be projected, B that through which it is to pass after being reflected from the planes CK , DH , EF in order.

Draw BG perpendicular to the plane EF , and take FG , such that $\frac{BF}{FG} = \lambda$; from G draw GI perpendicular to DH , so that $\frac{GH}{HI} = \lambda$; and

IL perpendicular to CK , so that $\frac{IK}{KL} = \lambda$; join AL , it is the direction

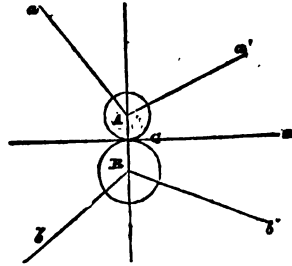
required. For join CI , DG , BE ; then from the last article it appears that after describing AC the body will be reflected in the direction CI ; and being incident upon the plane DH in the direction CD , it will be reflected in the direction DG ; and so from E to B .



OBLIQUE IMPACT.

48. PROB. *Two smooth bodies of given masses, moving in given directions with given velocities, strike against each other, it is required to find the velocity and direction of the motion of each of them after impact.*

Let A, B be the centres and C the point of contact of the two balls at the moment of impact. aA, bB the directions of their motions before impact, and Aa', Bb' their directions after impact. Cx the plane of impact, (Art. 37). The exact position of this plane can always be calculated from the known velocities and directions of the motions, and the known positions of the balls at a given time, before impact. We shall here suppose it given.



Let A, B be the masses of the balls; α, β the inclinations of aA, bB to the plane of impact; α', β' the inclinations of Aa', Bb' to the same plane.

Let a, b be the velocities before impact; and a', b' the velocities after impact.

Then the velocities of A and B parallel to Cx before impact are

$$a \cos \alpha, b \cos \beta \dots \dots \dots (\text{Art. 18, 3})$$

and their velocities perpendicular to Cx before impact are

$$a \sin \alpha, b \sin \beta \dots \dots \dots (\text{Art. 18, 4}).$$

As in the second Law of Motion, we are at liberty to consider each of these motions separately. The former will be unaffected by the impact, because the balls being smooth can offer no resistance in the direction of their common tangent Cx ; hence their velocities parallel to Cx will be the same after as before impact;

$$\therefore a' \cos \alpha' = a \cos \alpha \dots \dots \dots (1)$$

$$\text{and } b' \cos \beta' = b \cos \beta \dots \dots \dots (2).$$

The motions of the balls perpendicular to the plane of impact produce direct impact, and are altered by the impact; to apply Art. 39 to this case, we must write $a \sin \alpha, -b \sin \beta$ for a and b respectively; and $-a' \sin \alpha', b' \sin \beta'$ for a' and b' respectively; because in that Art. the velocities are estimated in the direction of A 's motion before impact:

$$\therefore a' \sin \alpha' = -\frac{Aa \sin \alpha - Bb \sin \beta}{A+B} + \lambda \cdot \frac{B(a \sin \alpha + b \sin \beta)}{A+B} \dots (3)$$

$$b' \sin \beta' = \frac{Aa \sin \alpha - Bb \sin \beta}{A+B} + \lambda \cdot \frac{A(a \sin \alpha + b \sin \beta)}{A+B} \dots (4).$$

These four equations furnish the complete solution of the problem: for (3) divided by (1), and (4) by (2) give $\tan \alpha'$ and $\tan \beta'$: and the square of (1) added to the square of (3) will give a' ; a similar process will give b' .

ON THE MOTION OF THE CENTRE OF GRAVITY OF A SYSTEM OF BODIES.

49. **PROB.** *Given the masses of several bodies, which move uniformly in straight lines, and the velocities and directions of their motions, and their positions at a certain time; to find the position of their common centre of gravity at any time, its path and the velocity and direction of its motion.*

1st. Let the motions of all the bodies take place in one plane, which take for the plane of xy ; let $m_1, m_2, m_3, \dots$ be their masses; $v_1, v_2, v_3, \dots$ their velocities; and $x'y', x''y'', x'''y''', \dots$ their co-ordinates at a given instant; $x_1y_1, x_2y_2, x_3y_3, \dots$ their co-ordinates at the time t after that instant: also, let $\alpha_1, \alpha_2, \alpha_3, \dots$ be the angles which the directions of their motions make with the axis of x .

At the time t the body m_1 has moved over the space $x_1 - x'$ parallel to the axis of x . But since its velocity in that direction is $v_1 \cos \alpha_1$ (Art. 18, 3) it must have moved over the space $v_1 \cos \alpha_1 t$ in that time (Art. 5);

$$\therefore x_1 - x' = v_1 t \cos \alpha_1;$$

$$\therefore x_1 = v_1 t \cos \alpha_1 + x'.$$

$$\text{Similarly } y_1 = v_1 \sin \alpha_1 t + y'.$$

These are the co-ordinates of m_1 at the time t : the co-ordinates of m_2 , may be determined from these by writing v_2, α_2, x'', y'' , for v_1, α_1, x', y' , and so on for the other bodies.

Now by the nature of the centre of gravity if $\bar{x}, \bar{y}$, be its co-ordinates at the time t , we have

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \dots}{m_1 + m_2 + m_3 + \dots}$$

or writing for $x_1, x_2, x_3, \dots$ their values before determined,

$$\begin{aligned} \bar{x} = & \left(\frac{m_1 v_1 \cos \alpha_1 + m_2 v_2 \cos \alpha_2 + m_3 v_3 \cos \alpha_3 + \dots}{m_1 + m_2 + m_3 + \dots} \right) . t \\ & + \frac{m_1 x' + m_2 x'' + m_3 x''' + \dots}{m_1 + m_2 + m_3 + \dots} \end{aligned}$$

This equation for brevity is written thus

$$\bar{x} = \frac{\Sigma (m v \cos \alpha)}{\Sigma (m)} . t + \frac{\Sigma (m x')}{\Sigma (m)} .$$

$$\text{Similarly } \bar{y} = \frac{\Sigma (m v \sin \alpha)}{\Sigma (m)} . t + \frac{\Sigma (m y')}{\Sigma (m)} .$$

These equations fully determine the position of the centre of gravity at any time t . From them also we find that the co-ordinates of this point when $t = 0$, were $\frac{\Sigma (m x')}{\Sigma (m)}$ and $\frac{\Sigma (m y')}{\Sigma (m)}$, and therefore the co-ordinates have increased respectively by $\bar{x} - \frac{\Sigma (m x')}{\Sigma (m)}$ and $\bar{y} - \frac{\Sigma (m y')}{\Sigma (m)}$ in the time t : hence these, or their respective equals

$$\frac{\Sigma (m v \cos \alpha)}{\Sigma (m)} . t, \text{ and } \frac{\Sigma (m v \sin \alpha)}{\Sigma (m)} . t$$

are the spaces described by the centre of gravity parallel to the axis of x and y , in the time t ; consequently the velocities of it in these directions, found by Art. 5, are

$$\begin{aligned} & \frac{\Sigma (m v \cos \alpha)}{\Sigma (m)} \text{ parallel to the axis of } x, \\ \text{and } & \frac{\Sigma (m v \sin \alpha)}{\Sigma (m)} \text{ parallel to the axis of } y. \end{aligned}$$

Hence if $\bar{v}$ be the velocity of the centre of gravity, and $\bar{\alpha}$ the angle which its path makes with the axis of x ,

the former of these expressions $= \bar{v} \cos \bar{\alpha}$, and the latter $= \bar{v} \sin \bar{\alpha}$;

$$\therefore \bar{v} \cos \bar{\alpha} = \frac{\Sigma (mv \cos \alpha)}{\Sigma (m)}$$

$$\text{and } \bar{v} \sin \bar{\alpha} = \frac{\Sigma (mv \sin \alpha)}{\Sigma (m)}.$$

From which we obtain

$$\tan \bar{\alpha} = \frac{\bar{v} \sin \bar{\alpha}}{\bar{v} \cos \bar{\alpha}} = \frac{\Sigma (mv \sin \alpha)}{\Sigma (mv \cos \alpha)}$$

$$\text{and } \bar{v}^2 = \left\{ \frac{\Sigma (mv \cos \alpha)}{\Sigma (m)} \right\}^2 + \left\{ \frac{\Sigma (mv \sin \alpha)}{\Sigma (m)} \right\}^2.$$

50. 2dly. When the motions of the bodies are not all in one plane, the system must be referred to three co-ordinate planes fixed in space. Let $x'y'z'$, $x''y''z''$, $x'''y'''z'''$... be the co-ordinates of the bodies at the given instant; $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$,... their co-ordinates at the time t after that instant; $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$,... the angles which the directions of their respective paths make with the co-ordinate axes of xyz .

Then in a similar manner to that before employed, we should find

$$x_1 = v_1 \cos \alpha_1 t + x'$$

$$y_1 = v_1 \cos \beta_1 t + y'$$

$$z_1 = v_1 \cos \gamma_1 t + z'$$

for the co-ordinates of m_1 ; and similar equations for those of the other bodies, at the time t .

Also, if $\bar{x} \bar{y} \bar{z}$, be the co-ordinates of the centre of gravity at the time t ,

$$\bar{x} = \frac{\Sigma (mv \cos \alpha)}{\Sigma (m)} \cdot t + \frac{\Sigma (mx')}{\Sigma (m)}$$

$$\bar{y} = \frac{\Sigma (mv \cos \beta)}{\Sigma (m)} \cdot t + \frac{\Sigma (my')}{\Sigma (m)}$$

$$\bar{z} = \frac{\Sigma (mv \cos \gamma)}{\Sigma (m)} \cdot t + \frac{\Sigma (mz')}{\Sigma (m)}.$$

And if the path of the centre of gravity make angles $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ with the co-ordinate axes, and if $\bar{v}$ be its velocity,

$$\bar{v}^2 = \left\{ \frac{\Sigma (mv \cos \alpha)}{\Sigma (m)} \right\}^2 + \left\{ \frac{\Sigma (mv \cos \beta)}{\Sigma (m)} \right\}^2 + \left\{ \frac{\Sigma (mv \cos \gamma)}{\Sigma (m)} \right\}^2$$

$$\bar{v} \cos \bar{\alpha} = \frac{\Sigma (mv \cos \alpha)}{\Sigma (m)}$$

$$\bar{v} \cos \bar{\beta} = \frac{\Sigma (mv \cos \beta)}{\Sigma (m)}$$

$$\bar{v} \cos \bar{\gamma} = \frac{\Sigma (mv \cos \gamma)}{\Sigma (m)}.$$

The first of these gives $\bar{v}$ and then the last three give $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$.

It appears that the centre of gravity moves with uniform velocity in a straight line (Art. 20), and the equations of its path may be obtained by eliminating t between the equations which give the values of $\bar{x}$, $\bar{y}$, in Art. 49, or $\bar{x} \bar{y} z$ in this Article.

Hence, in Art. 49, we have

$$\frac{\bar{x} \Sigma (m) - \Sigma (mx')}{\Sigma (mv \cos \alpha)} = \frac{\bar{y} \Sigma (m) - \Sigma (my')}{\Sigma (mv \sin \alpha)}$$

for the equation of the path of the centre of gravity when the bodies move in one plane.

51. PROP. *If any of the bodies in the last Problem happen to impinge against each other in the course of their motions, the motion of the centre of gravity will not be affected thereby either in velocity or direction.*

1st. As to its velocity.

The velocity of the centre of gravity in the direction of x is

$$\frac{\Sigma (mv \cos \alpha)}{\Sigma (m)}.$$

The denominator of this fraction is constant; and the numerator is equal to the sum of the momenta of all the

bodies estimated parallel to the axis of x . Now it has been shewn in Art. 27, that whatever momentum is lost in impact by one body, is exactly gained by the other body against which it strikes; hence the whole quantity of momentum cannot be altered by impact, and therefore $\Sigma(mv \cos \alpha)$ remains constant, and consequently the velocity of the centre of gravity, parallel to the axis of x , is the same after impact as before; the same may be proved for the velocities parallel to the other axes; and therefore the whole velocity is unaltered by impact.

2dly. As to its direction.

The direction of the motion of the centre of gravity, that is, the inclinations of its path to the co-ordinate axes are given in terms of its velocities parallel to those axes (Art. 18, or the equations of Arts. 49, 50): and these having been just shewn to be the same after as before impact, the others remain constant; and so the centre of gravity moves on in the same line as before impact.

52. It has been shewn that the velocity of the centre of gravity in a direction parallel to the axis of x is $\frac{\Sigma(mv \cos \alpha)}{\Sigma(m)}$.

Now, if a body, equal to all the bodies taken together ($= \Sigma m$), were to move with the same velocity as the centre of gravity of the system, its momentum parallel to the axis of x would $= \Sigma(m) \cdot \frac{\Sigma(mv \cos \alpha)}{\Sigma(m)} = \Sigma(mv \cos \alpha)$ = the whole momentum of the system in that direction. The same may be proved for the momenta parallel to the other axes.

53. Remark. Conversely, if the whole momentum of the system were communicated to a body equal to the sum of all the bodies, that body would move with the same velocity, and in the same direction as the centre of gravity of the system.

54. Remark. Two consecutive points are necessary and sufficient to determine the velocity and direction of a body's motion. For an indefinitely short period therefore each body, even if its path be curvilinear, may be considered as moving uniformly in a straight line: consequently the

five preceding articles may be applied to the instantaneous state of *any* system, whatever be the velocities and paths of the particles. In using those results, we must observe that $v_1, v_2, v_3, \dots$ denote the velocities of the particles at any proposed instant; and $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3, \dots$ the inclinations of the tangents to their paths at the same moment to the axes of x, y, z . $\bar{v}_1, \bar{\alpha}, \bar{\beta}, \bar{\gamma}$ will give the contemporaneous velocity and direction of the motion of the centre of gravity. Its path will be found, as before, by eliminating t ; but it will not necessarily be a straight line.

55. *PROB. Two spherical bodies move from given positions in given straight lines with given velocities; to find when their centres will be a given distance apart.*

Let $x', y', z', x'', y'', z''$ be the co-ordinates of the centres of the balls at a given instant; $x_1, y_1, z_1, x_2, y_2, z_2$ their co-ordinates at the time t from that instant, when their distance apart is equal to D . Let v_1, v_2 be their velocities; and $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ the angles which their paths make with the co-ordinate axes.

$$\text{Then } D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \dots (1).$$

But, as in Art. 49,

$$x_1 = v_1 \cos \alpha_1 \cdot t + x'$$

$$x_2 = v_2 \cos \alpha_2 \cdot t + x'';$$

$$\therefore x_1 - x_2 = (v_1 \cos \alpha_1 - v_2 \cos \alpha_2) t + (x' - x'').$$

$$\text{Similarly, } y_1 - y_2 = (v_1 \cos \beta_1 - v_2 \cos \beta_2) t + (y' - y''),$$

$$\text{and } z_1 - z_2 = (v_1 \cos \gamma_1 - v_2 \cos \gamma_2) t + (z' - z'').$$

And these being substituted in the above equation (1) will give a quadratic equation for determining the value of t . This equation will have two roots, and therefore shews that in general the balls are twice at a given distance apart, viz. once before being at their least distance and once after.

To find the least distance to which the balls can approach each other, we must put the differential coefficient of D , with respect to t , equal to zero; which will give a

simple equation for finding t , the time of being at a minimum distance; and this value of t being written in the equation (1) will give the minimum distance itself. If this value of D should be *less than* the sum of the radii of the balls, they will have impinged upon each other before coming to that distance; and so the least practical distance is, in this case, the sum of their radii. And to find the time of impact, we must write the sum of the radii for D in (1), and solve the quadratic in t , the *least* root thus found will be the time of impact. The greater root refers to the time when the surfaces of the balls would be a second time just in external contact, supposing the substance of each ball to be penetrable to the other, so that no *impact* might take place upon their first meeting. As this, though algebraically a possible problem, is physically impossible, on account of the impenetrability of matter, the greater root is to be rejected.

CHAPTER III.

ON UNIFORM ACCELERATING OR RETARDING FORCES.

56. PROP. *If a body be acted on in the line of its motion by a uniformly accelerating force f , the velocity generated in the time $t = ft$.*

For, because the force is uniform (Art. 11), there is an additional velocity equal to f generated in each successive unit of time; and therefore the velocity generated in t units of time is equal to ft .

57. What has just been proved of velocity generated, is equally true of velocity destroyed by a retarding force. And consequently when a body is under the influence of an uniform force, the velocity gained or lost, in passing from one point to another of its path, divided by the force, gives the time of motion. This is to be understood as applying only to velocity generated or destroyed in the direction in which the force acts: and does not apply to motion which the body may have in any other direction.

58. Hence if at the time zero, u be the known velocity of a body, which is acted on by a uniform force, its velocity at the time t will be

$$= u + ft, \text{ if the force accelerates the motion}$$

$$\text{or } = u - ft, \text{ if retards}$$

59. *A body at rest is acted on by a given uniform accelerating force f , to find the space through which it will move in a given time.*

Let s be the space moved through in the time t from the beginning of the motion.

Divide the time of motion into n equal intervals, each equal to $\frac{t}{n}$; then the velocities which the body has acquired at the *end* of the 1st, 2nd, 3rd,... and last of these intervals, by Art. 56, are respectively

$$f \cdot \frac{t}{n}, f \cdot \frac{2t}{n}, f \cdot \frac{3t}{n}, \dots f \cdot \frac{nt}{n}.$$

Now if the body had moved uniformly during each interval with the velocity which it has at the *end* of the respective intervals, it would have described a greater space in the time t than in reality it does describe: this is evident from the consideration, that upon the supposition here made, the velocity of the body would always be too great, except at the very end of each interval. But the spaces that would be described, upon this hypothesis, during the 1st, 2nd, 3rd,... and last intervals, by Art. 5, are respectively

$$f \cdot \frac{t^2}{n^2}, f \cdot \frac{2t^2}{n^2}, f \cdot \frac{3t^2}{n^2}, \dots f \cdot \frac{nt^2}{n^2};$$

$$\therefore s \text{ cannot be } > f \frac{t^2}{n^2} + f \frac{2t^2}{n^2} + f \frac{3t^2}{n^2} + \dots + f \frac{nt^2}{n^2}$$

$$> \frac{ft^2}{n^2} (1 + 2 + 3 + \dots + n)$$

$$> \frac{ft^2}{n^2} \cdot \frac{n(n+1)}{2}$$

$$> \frac{ft^2}{2} \cdot \left(1 + \frac{1}{n}\right) \dots \dots \dots (1).$$

Again, the velocities which the body has at the *beginning* of each interval are respectively for the 1st, 2nd, 3rd,... and last interval

$$0, f \frac{t}{n}, f \frac{2t}{n}, \dots f \frac{(n-1)t}{n},$$

and if it had moved uniformly during the respective intervals with these velocities, it would have described spaces equal to

$$0, f \frac{t^2}{n^2}, f \frac{2t^2}{n^2}, \dots f \frac{(n-1)t^2}{n^2},$$

each being respectively smaller than the actual space, because on the hypothesis now made, the velocity of the body is always too small except at the very beginning of each interval.

Hence

$$\begin{aligned} s \text{ cannot be } &< 0 + f \frac{t^2}{n^2} + f \frac{2t^2}{n^2} + \dots + f \frac{(n-1)t^2}{n^2} \\ &< \frac{ft^2}{n^2} \{1 + 2 + \dots (n-1)\} \\ &< \frac{ft^2}{n^2} \cdot \frac{n(n-1)}{2} \\ &< \frac{ft^2}{2} \cdot \left(1 - \frac{1}{n}\right) \dots \dots \dots (2). \end{aligned}$$

We have thus found two quantities (1) and (2), such that s cannot be greater than the former nor less than the latter, whatever be the number of intervals into which the time of motion is divided. Let the number of intervals be increased without limit, on which supposition $\frac{1}{n}$ vanishes from the expressions (1) and (2) and each becomes equal to $\frac{ft^2}{2}$. And this quantity is such that s is neither greater nor less than it; consequently

$$s = \frac{1}{2} ft^2.$$

60. The velocity acquired by the body in last article equals ft ; if this be called v , and the equation $v = ft$ be united with $s = \frac{1}{2} ft^2$, we shall obtain the relations which exist between force, velocity, time and space, when the motion

is due to the action of a uniform accelerating force. These are

$$v = ft, \quad s = \frac{1}{2}ft^2, \quad s = \frac{1}{2}vt, \quad \text{and} \quad v^2 = 2fs,$$

which the reader will find it useful to remember; and to notice that they are only applicable when the body moves from rest.

Remark. If $t = 1$, $f = 2s$, so that the accelerating force is measured by twice the space through which it will draw the body from rest in a unit of time.

61. *If the whole time of motion be divided into equal intervals, the spaces described in these successive intervals are proportional to the odd numbers 1, 3, 5, 7.....*

For, let τ be the length of each interval; then the time elapsed from the beginning of the motion to the end of the $(r)^{\text{th}}$ interval is $r\tau$, and the whole space described up to that moment $= \frac{1}{2}f\tau^2r^2$. Similarly the space described up to the end of the $(r-1)^{\text{th}}$ interval is $= \frac{1}{2}f(r-1)^2\tau^2$, the difference of these, or the space described in the r^{th} interval $= \frac{1}{2}f\tau^2r^2 - \frac{1}{2}f(r-1)^2\tau^2 = \frac{1}{2}f\tau^2(2r-1)$. And by giving to r the successive values 1, 2, 3, 4... we have

$$\text{space described in 1st interval} = \frac{1}{2}f\tau^2.1$$

$$\dots\dots\dots \text{2nd} \dots\dots\dots = \frac{1}{2}f\tau^2.3$$

$$\dots\dots\dots \text{3rd} \dots\dots\dots = \frac{1}{2}f\tau^2.5$$

$$\dots\dots\dots \text{4th} \dots\dots\dots = \frac{1}{2}f\tau^2.7$$

&c. = &c.

which are as the numbers 1, 3, 5, 7.....

62. The last three articles apply exclusively to the case of a body moving from *rest*, by the action of a uniform accelerating force. If in Art 59, the body had had an original velocity (u) of projection in the line in which the force acts, the necessary modification of the resulting expression between s and t may be effected by the assistance of the second Law of Motion. For if no force had acted, the body would have moved through the space ut , in the time t ; and if there had been no velocity of projection, the body would have been drawn through the space $\frac{1}{2}ft^2$

in that time: these effects are independent of each other by the second Law of Motion; and therefore the whole effect is found by uniting these two effects.

Hence $s = ut + \frac{1}{2}ft^2$ when the force *accelerates* the motion,
and $= ut - \frac{1}{2}ft^2$ *retards*

63. Let a body placed at A be drawn by the action of a uniform accelerating force f from A to C in any time; and suppose B any point in AC .

Then, by Art. 60,

$$(\text{vel.})^2 \text{ at } C = 2f.AC,$$

$$(\text{vel.})^2 \text{ at } B = 2f.AB;$$

$$\therefore (\text{vel.})^2 \text{ at } C - (\text{vel.})^2 \text{ at } B = 2f.BC;$$

$$\therefore (\text{vel.})^2 \text{ at } C = (\text{vel.})^2 \text{ at } B + 2f.BC$$

$$= (\text{vel.})^2 \text{ at } B + (\text{vel.})^2 \text{ through } BC \text{ from rest at } B.$$

[$\begin{matrix} A \\ B \\ C \end{matrix}$

DEF. The velocity which a body would acquire in falling from rest through a certain space by the action of a uniform force, is called "the velocity *due* to that space;" and conversely, the space through which a body must fall from rest to acquire a certain velocity by the action of a given uniform force, is called "the space *due* to that velocity."

Hence the property expressed by the above equation may be thus enunciated;

The square of the velocity due to any space is equal to the sum of the squares of the velocities due to any two parts into which that space is divided.

64. *The Earth exerts upon terrestrial bodies a uniformly accelerating force, which is the same for all substances.*

1. The accelerating force which the Earth exerts is the same for *all substances* at a given place.

This seems to be contradicted by daily observation of bodies falling in the air, which would rather incline us to the supposition that heavy substances descend to the ground in less time than light ones: but this difference can easily be shewn

to arise entirely from the resistance which the air offers to bodies moving within it. For to take an extreme case, if a feather and a piece of lead be let fall at the same moment from the top of the exhausted receiver of an air pump, they will be found to strike the bottom at the same moment. The same experiment has often been carefully tried with other substances, and always with the same result. Consequently at the same place the Earth exerts the same accelerating force on *all* bodies.

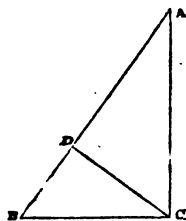
2. This accelerating force is *uniform*.

If this be the case it appears from the expression $s = \frac{1}{2}ft^2$ (Art. 59) that the spaces through which a body will fall from rest in different times are exactly proportional to the squares of the times of falling. But the motion of falling bodies is much too quick to admit of this experiment being made with such exactness as the importance of the subject demands. In rapid motions the air's resistance is too great to be neglected; and therefore the formula $s \propto t^2$ cannot be verified by bodies falling freely. To overcome this difficulty, two unequal bodies are suspended by a string over a fixed pulley, means being used to diminish the friction of the pulley as much as possible. A machine on this principle, fitted up with apparatus for measuring spaces descended and for accurately marking the moment of reaching a given point of the descent, and with a clock for noting time, was used by Mr. Attwood with perfect success in verifying the formula $s \propto t^2$. By diminishing the inequality of the weights employed, the motion may be rendered as slow as the experimenter may require. In this way it has been fully proved that a body falling freely at London acquires in every second of its descent an additional velocity of 32.18 feet per second.

Hence the accelerating force of gravity is uniform, and is the same for all substances. It is usually denoted by the letter g (Art 25); and in the latitude of London $g = 32.18$ feet.

65. PROP. *If a body descend down a smooth inclined plane by the action of gravity, the force which accelerates the motion is uniform and equal to g multiplied by the sine of the plane's inclination to the horizon.*

Let AB be an inclined plane; draw AC , BC perpendicular and parallel to the horizon; and CD perpendicular to AB . Take AC to represent g the force of gravity; and resolve it into two forces AD , DC , which are equivalent to g . Then by the second Law of Motion we may consider the effects of these separately.



The latter DC , which $= g \cos ACD = g \cos B$, being always perpendicular to the plane, would produce motion in the direction DC , but as this is prevented by the impenetrability of the plane, it produces no effect at all upon the motion of the body.

The former AD , which $= g \sin ACD = g \sin B$, acting in the direction AB is that which alone produces the motion along the plane, and since the plane is quite smooth, this force is unresisted and produces its full effect: and as it is constant, the proposition to be proved is made out in both parts.

Remark. As the force which urges a body down an inclined plane is uniform, the propositions from Art. 56 to Art. 63, will be applicable to motion on inclined planes by writing $g \sin B$ for f . Also, for bodies descending on inclined planes from rest, space $\propto$ (time)²; and therefore the constancy and uniformity of the force of gravity may be proved by the descent of bodies down inclined planes.

66. PROP. *The velocity acquired by a body falling down an inclined plane is equal to that which it would acquire in falling down its perpendicular altitude; whether the body begin to fall from rest or be projected with a given velocity.*

Let AB be the inclined plane, and AC its perpendicular altitude.

1st. When the body begins to fall from rest.

By the equation in Art 60, when the body falls down the inclined plane,

$$\begin{aligned}
 (\text{vel.})^2 \text{ acquired down } AB &= 2g \sin B \cdot AB \\
 &= 2g \cdot AC \\
 &= (\text{vel.})^2 \text{ acquired down } AC; \\
 \therefore \text{vel. acquired down } AB &= \text{vel. acquired down } AC.
 \end{aligned}$$

2nd. When the body is projected with a given velocity u .

By Art. 63, we have for the motion down the plane,

$$\begin{aligned}
 (\text{vel.})^2 \text{ at } B &= (\text{vel.})^2 \text{ at } A + (\text{vel.})^2 \text{ due to } AB \\
 &= u^2 + 2g \sin B \cdot AB \\
 &= u^2 + 2g \cdot AC.
 \end{aligned}$$

For the motion down the altitude AC ,

$$\begin{aligned}
 (\text{vel.})^2 \text{ at } C &= (\text{vel.})^2 \text{ at } A + (\text{vel.})^2 \text{ due to } AC \\
 &= u^2 + 2g \cdot AC \\
 &= (\text{vel.})^2 \text{ at } B; \\
 \therefore \text{vel. at } C &= \text{vel. at } B, \\
 \text{and vel. at } A &= \text{vel. at } A.
 \end{aligned}$$

Hence by subtracting the latter from the former of the last two equations,

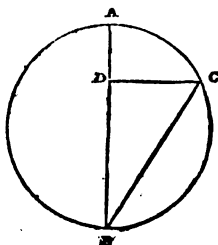
$$\text{vel. acquired down } AC = \text{vel. acquired down } AB.$$

67. PROP. *If chords be drawn from either extremity of a vertical diameter of a circle, the time of descent down each of them will be equal to the time down the diameter.*

For let AB be a vertical diameter and BC any chord. Draw CD perpendicular to AB . Then BCD equals the inclination of the plane BC to the horizon: hence the accelerating force down CB

$$= g \sin BCD = g \cdot \frac{BD}{BC} = g \cdot \frac{BC}{AB};$$

and by Art. 59, space = $\frac{1}{2}$. force $\times$ (time)², we have



$$BC = \frac{1}{2}g \cdot \frac{BC}{AB} \cdot (\text{time down } CB)^2;$$

$$\therefore (\text{time down } CB)^2 = \frac{2AB}{g}.$$

By the same formula, when the body falls down the diameter AB ,

$$AB = \frac{1}{2}g \cdot (\text{time down } AB)^2;$$

$$\therefore (\text{time down } AB)^2 = \frac{2AB}{g} = (\text{time down } CB)^2;$$

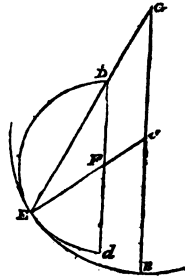
$$\therefore \text{time down } AB = \text{time down } CB,$$

and as CB is any one of the chords the proposition is proved.

68. The property of the circle just demonstrated is very useful in solving geometrically all problems concerning planes of quickest and slowest descent; as the following example will shew.

PROB. To find the plane down which a body will descend in the least time from a point within a vertical circle to its circumference.

Let D be the given point; C the centre of the vertical circle BE ; BCG the vertical diameter. Join GD and produce it to meet the circumference in E ; DE is the plane required. For, join CE ; and draw DF parallel to BG . Then, because $EC = CG$; therefore $EF = FD$, and a circle described with centre F and radius FE will pass through E and D , and touch the given circle at E , because CE which passes through their centres must pass through the point of contact.

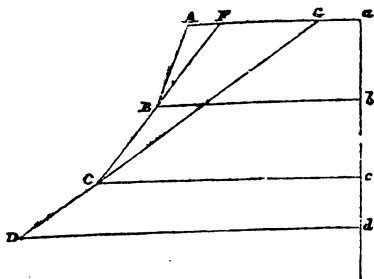


Now, if any plane (except DE) be drawn from D to the given circle, it will be partly without the circle dED , and be longer than the part of it which forms a chord to the

circle drawn from the extremity D of a vertical diameter. But the time down this part is equal to the time down DE , each being equal to the time down Dd (Art. 67); hence the time down DE is less than the time down the whole of that plane of which the chord is but a part: and as this is any plane drawn from D , except DE ; therefore the time down DE is the least.

69. PROP. *If a body, by the action of gravity, descend along a smooth continuous curve line situated in a vertical plane, its velocity at any point will be the same as if it had fallen freely in a straight line through the same vertical space.*

Because the curve line is smooth, it can offer no resistance to the motion of the body along it, and therefore no velocity will be lost through the action of the curve. Now, instead of the curve, substitute smooth planes AB , BC , CD , such that when their number is increased and the length of each diminished without limit, the system of planes shall coincide with the curve line. As no velocity is lost by the action of the curve, we may *ab initio* introduce the supposition that no velocity is lost by the action of the planes, as a body passes from one of them to the next: for though this hypothesis is not true (See Art. 45) as long as the number of planes is finite, we have shewn it is true when their number is infinite; consequently the error committed by supposing no velocity lost in passing from plane to plane will vanish when we make the number of them infinite; that is, there will be no error in the reasoning that will affect our ultimate inference with regard to the motion on the curve line. This being premised, draw the horizontal lines Aa , Bb , Cc ,... and the vertical line $abcd$... also produce CB , DC ,... to F , G ... Then, whatever be the velocity of pro-



jection down AB , the velocity acquired through AB equals that acquired through ab (Art. 66), equals that acquired through FB . Hence, since no velocity is supposed to be lost in passing from AB to BC , the body begins to descend down BC under the same circumstances as if it had descended down FB . Hence the velocity acquired at C , equals that acquired through FC , equals that acquired through ac , equals that acquired through GC . Hence the velocity at D equals that at d . This process may be continued to any number of planes, and therefore the same result follows when the number is infinite, and the system of planes becomes the curve line, in which the same is true. Hence, *if a body, &c.*

70. If the body had been projected up the curve line, the same proposition is true, with the requisite verbal alterations; thus, *the velocity at any point, will be the same as if it had been projected vertically upwards, and had ascended through the same vertical space.* It therefore appears, that whether a body ascend or descend along a smooth curve line, the velocity depends not at all on the form of the curve, but simply on the vertical space ascended or descended. Hence we can prove, that if a body acquire velocity by descending down a curve line, it will lose it again by ascending along any other curve line to the same vertical height as that from which it fell. To establish this, we have only to shew that, in falling down a vertical line, a body gains as much velocity as it loses in ascending through the same. Let a body be projected downwards from B towards C , with a velocity u : and let v be its velocity at C . Then, by Art. 63,

$$(\text{vel.})^2 \text{ at } C = (\text{vel.})^2 \text{ at } B + 2g \cdot BC,$$

$$\text{or } v^2 = u^2 + 2g \cdot BC \dots \dots (1).$$

Let now the body be projected upwards from C with the velocity v , and suppose t the time of ascending to B , and u' its velocity when it reaches B .

Then, by Art. 62,

$$BC = vt - \frac{1}{2}gt^2 \dots \dots (2),$$

and by Art. 58,

$$\begin{aligned}
 u' &= v - gt; \\
 \therefore u'^2 &= v^2 - 2vgt + g^2t^2 \\
 &= v^2 - 2g(vt - \tfrac{1}{2}gt^2) \\
 &= v^2 - 2g \cdot BC \text{ from (2)} \\
 &= u^2 \text{ from (1);} \\
 \therefore u' &= u.
 \end{aligned}$$

71. If C be the lowest point of a curve line ACB , of which the two branches CA , CB have a common tangent at C and are exactly similar and equal; a body let fall from A will ascend to B ; and, since the velocities are equal at all equal altitudes in the ascent and descent, they are equal at those points of the two branches which are similarly situated; and consequently, if the two branches be divided into the same number of corresponding indefinitely small arcs, the arcs of each pair will be described with equal velocities, and, being equal in length, in equal times. Hence the time of descending through an indefinitely small arc in one branch, is equal to the time of ascending through the corresponding portion of the other branch. And this being true of each pair of indefinitely small arcs, will be true of the whole arcs; that is, the time of descending through AC is equal to the time of ascending through CB .

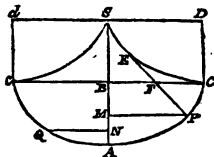


72. Remark. Our propositions respecting the motion of bodies upon curve lines have been proved on the supposition of the curve lines being *perfectly smooth*; which amounts to supposing the curve's resistance to be offered only in the direction of a normal at the point of contact with the moving body. If any means can be found of retaining this normal force without introducing resistance in the tangential direction when the curve line is removed, the same propositions will be true. In some cases this may be effected very readily. Thus, if a body be suspended by a very fine string from a fixed point; when it is raised through any angle less than 90° , and let go, it will oscillate about the fixed point in a circular arc; the tension of the string

which retains it in the arc being a normal force. In other cases, the end can only be attained by more complex means, as in the following example.

PROB. *To make a body oscillate in a given cycloid.*

Let $CPAc$ be the given cycloid, its plane being vertical; AB its axis, Cc its horizontal base. Produce AB to S , making $BS = AB$: complete the parallelogram Cd ; and upon SD , Sd as bases, and CD , cd as axes, describe semi-cycloids SC , Sc , which, having equal axes and bases to the given one, will be equal to it in all respects.



In most treatises on the *Differential Calculus* it is shewn that the two semi-cycloids CS , Sc are the evolute of CAC ; and the general property of evolutes (that indeed from which they derive their name) is, that if a string be wrapped upon the evolute CS , and made fast at S , and then unwound, its extremity will trace out the curve CA ; and as it is afterwards wound upon Sc , the same extremity will trace out the other half cycloid Ac ; and in all this motion, the unwound part EP in any position of the string SEP , will be a normal to the curve at P .

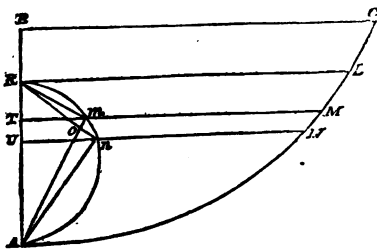
Hence, if a body P were suspended by such a string, it would oscillate in the manner required.

73. PROB. *To find the time of oscillation in a cycloidal arc.*

Let CA be the half of the cycloid in which the body oscillates, its axis AB being vertical, and base CB horizontal. Let L be the point from which the body begins to descend. Then the time of a complete oscillation, which equals the time of descent + time of ascent, is equal to twice the time of descending through LA (Art. 71). Divide the arc LA into very small parts, of which let MN be any one. Draw the horizontal lines LR , MT , NU ; and upon AR describe a semicircle cutting MT in m , and NU in n ; join Am , An , Rm , and Rn . Then the velocity of the body when at M , by Art. 69, is that due to the vertical space RT ;

$$\therefore \text{vel. at } M = \sqrt{2g \cdot RT} \dots \text{Art. 60.}$$

And when the number of parts into which LA is divided is increased, and the length of each diminished without limit, MN will be so small that the velocity of the body in moving through it may be considered uniform, in which case (by Art. 5)



$$\text{the time of moving from } M \text{ to } N = \frac{\text{space}}{\text{vel.}} = \frac{MN}{\sqrt{2g \cdot RT}}.$$

An arc of a cycloid, measured from its lowest point, is equal to twice the length of the corresponding chord of the generating circle. This is proved in most treatises on the Integral Calculus.

Now AB is equal to the diameter of the generating circle, and hence

$$MN = AM - AN$$

$$= 2 \cdot \text{chord of circle corresponding to } AM - 2 \cdot \text{chord}$$

$$\text{corresponding to } AN$$

$$= 2\sqrt{AB \cdot AT} - 2 \cdot \sqrt{AB \cdot AU},$$

by the nature of the circle.

Hence the time of moving through MN ,

$$\begin{aligned} &= \frac{2\sqrt{AB \cdot AT}}{\sqrt{2g \cdot RT}} - \frac{2\sqrt{AB \cdot AU}}{\sqrt{2g \cdot RT}} \\ &= \frac{2\sqrt{AB}}{\sqrt{2g}} \cdot \left\{ \sqrt{\frac{AT}{RT}} - \sqrt{\frac{AU}{RT}} \right\} \\ &= \sqrt{\frac{2AB}{g}} \cdot \left\{ \sqrt{\frac{AT \cdot AR}{RT \cdot AR}} - \sqrt{\frac{AU \cdot AR}{RT \cdot AR}} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{\frac{2AB}{g}} \cdot \left\{ \frac{Am}{Rm} - \frac{An}{Rm} \right\} \\
 &= \sqrt{\frac{2AB}{g}} \cdot \frac{mo}{Rm} * \\
 &= \sqrt{\frac{2AB}{g}} \cdot \angle mRn.
 \end{aligned}$$

The same may be proved for the time through each of the other parts into which LA was divided; hence adding all these times together, we have the time from L to A ,

$$\begin{aligned}
 &= \sqrt{\frac{2AB}{g}} \cdot \left\{ \begin{array}{l} \text{Sum of all the angles subtended at } R \text{ by} \\ \text{the parts, such as } mn, \text{ into which the} \\ \text{semicircle upon } AR \text{ is divided.} \end{array} \right\} \\
 &= \sqrt{\frac{2AB}{g}} \cdot \left\{ \begin{array}{l} \text{Angle subtended at } R \text{ by the whole semi-} \\ \text{circle } RmnA. \end{array} \right\} \\
 &= \sqrt{\frac{2AB}{g}} \cdot \frac{\pi}{2}.
 \end{aligned}$$

Hence the time of making a complete oscillation

$$= \pi \sqrt{\frac{2AB}{g}},$$

which is the same whether the body oscillate through a large or small arc of the cycloid.

74. By the property of the cycloid, $2AB$ is equal to the length of the string SA mentioned in Article 72. Now since a circle whose radius is SA (Fig. of Art. 72) and centre S , coincides with the cycloid at and very near to A , being in fact the circle of curvature at A , the oscillation in a

* We here assume that $Am - An = mo$, which may be thus shewn. The angle $AOR = AnR + mAn$; now mAn vanishes ultimately because mn is indefinitely small; therefore ultimately $AOR = AnR =$ a right angle: therefore $Aon = AnO$ and $An = Ao$. In the next line we assume $\frac{mo}{Rm} = \sin mRo =$ angle $mRo =$ angle mRn ; because the arc = its sine, ultimately.

very small arc of this circle will be executed in the same manner and time as in a very small cycloidal arc. Hence if a body oscillate in a small circular arc, the radius of which, or length of the string by which it is suspended, is equal to L feet, the time of making an oscillation is equal to

$$\pi \sqrt{\frac{L}{g}}.$$

75. By the assistance of the last article we may solve all problems relating to the times of oscillation of clock pendulums.

I. *If at a given place the pendulums of two clocks be of different lengths, their times of oscillation are proportional to the square roots of their respective lengths; and the number of oscillations which they make in the same time are inversely as the square roots of their lengths.*

For let L, L' be their lengths, then, g being the same for both,

time of one oscil. of the former : time of one oscil. of the latter,

$$\begin{aligned} &:: \pi \sqrt{\frac{L}{g}} : \pi \sqrt{\frac{L'}{g}}, \\ &:: \sqrt{L} : \sqrt{L'}. \end{aligned}$$

Again, the number of oscillations in a given time for the former : the number in same time for the latter

$$:: \frac{\text{the given time}}{\text{time of one oscil.}} \text{ in the former} : \frac{\text{the given time}}{\text{time of one oscil.}} \text{ in the latter}$$

$$:: \frac{1}{\text{time of one oscil.}} \text{ in the former} : \frac{1}{\text{time of one oscil.}} \text{ in the latter}$$

$$:: \frac{1}{\sqrt{L}} : \frac{1}{\sqrt{L'}}.$$

II. *If the same pendulum be made to oscillate at different places, the number of oscillations made in a day, or any given time, will be as the square root of the force of gravity at the respective places.*

For L is the same at the two places; let g, g' be the force of gravity. Then

time of one oscillation at former place : ditto at the latter

$$\therefore \pi \sqrt{\frac{L}{g}} : \pi \sqrt{\frac{L}{g'}},$$

$$\therefore \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{g'}}.$$

Hence the number of oscillations in a given time in the former case : number in the latter

$$\therefore \frac{1}{\text{time of one oscil.}} \text{ in the former : ditto in the latter}$$

$$\therefore \sqrt{g} : \sqrt{g'}.$$

By this we see that the pendulum may be made use of in comparing the force of the Earth's attraction at different places. And we shall shew hereafter how it may be made use of in finding the actual value of g ; and in proving that g is constant for small altitudes above the Earth's surface.

III. *The length of a second's pendulum being slightly altered by a change of temperature or other means; to find the daily error of the clock.*

Let l be the length of the second's pendulum, $l + \delta l$ be the length of the same pendulum when changed by temperature: N the number of seconds in one day equal 86400.

$$\text{Then} \quad 1 = \pi \sqrt{\frac{l}{g}} \dots\dots\dots (\text{Art. 74}),$$

and the time of one oscillation of the pendulum whose length is $l + \delta l$,

$$\begin{aligned} &= \pi \sqrt{\frac{l + \delta l}{g}} \\ &= \pi \sqrt{\frac{l}{g}} \cdot \left(1 + \frac{\delta l}{l}\right)^{\frac{1}{2}} = \left(1 + \frac{\delta l}{l}\right)^{\frac{1}{2}}. \end{aligned}$$

Hence the number of seconds (or of oscillations) lost in one day

$$\begin{aligned}
 &= N - \frac{N}{\left(1 + \frac{\delta l}{l}\right)^{\frac{1}{2}}} \\
 &= N - N \left(1 - \frac{\delta l}{2l}\right) \text{ neglecting } (\delta l)^2, (\delta l)^3, \dots \\
 &= \frac{N}{2} \cdot \frac{\delta l}{l}.
 \end{aligned}$$

IV. *A clock loses n seconds per diem, to find how much the pendulum must be shortened to correct it.*

Let δl be the quantity by which it must be shortened. Then by the last article

$$\begin{aligned}
 n &= \frac{N}{2} \cdot \frac{\delta l}{l}; \\
 \therefore \delta l &= \frac{2n}{N} \cdot l.
 \end{aligned}$$

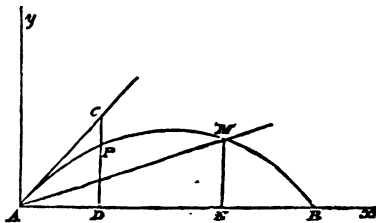
MOTION OF PROJECTILES IN VACUO.

In the preceding Articles of this Chapter, we have determined the motion of bodies, subject to the influence of gravity, both when they were suffered to move freely in a vertical direction, or by some mechanical contrivance constrained to move upon a plane or in a curve line. It yet remains to determine a species of motion which belongs to neither of these cases; viz. that of a body thrown or projected, in an oblique direction, and left without constraint to choose its own path. Such is the motion of a stone thrown at a distant mark. It may be here mentioned, that this kind of motion, as well as those before considered, is supposed to take place *in vacuo*; an hypothesis, which is introduced solely for the purpose of simplifying our calculations. Now, as all motion near the Earth's surface is impeded by the resistance of the atmosphere, our results will differ from experiment; but, as the resistance

of the air is but small, when the motion is not very quick, the error in such case will not amount to much. We shall shew hereafter how to effect the proper corrections for this error. From what is here stated, the reader will perceive, that besides determining the time and velocity of the body's motion, we have now to investigate the form and magnitude of the path which it will take in its flight.

76. **PROB.** *A body being projected from a given point in a given direction and with a given velocity; it is required to find the place of the body at any time, and the equation of the path which it will describe.*

Let A be the point from which the body is projected, and AC the direction of projection. Draw Ax horizontal, and Ay vertical, which take for the axes of x and y ; and suppose the curve line APB to be the path which the body will describe; and suppose the body to be at P at the time t from the moment of projection. Draw CPD vertical.



$AD = x$, $DP = y$, $\alpha = BAC$, and u = the velocity of projection.

Now the actual motion is compounded of two motions, the original velocity in the direction AC , and that which is generated by the force of gravity. By the second Law of Motion we are at liberty to calculate the effect of gravity separately. The projectile velocity in the time t would carry the body from A to C , such that $AC = ut$; and the force of gravity would then cause it to descend from C to P in the same time t , such that $CP = \frac{1}{2}gt^2$. Hence

$$x = AD = AC \cos \alpha = ut \cos \alpha \dots \dots \dots (1),$$

$$\text{and } y = CD - CP = AC \sin \alpha - \frac{1}{2}gt^2$$

$$= ut \sin \alpha - \frac{1}{2}gt^2 \dots \dots \dots (2).$$

These two equations give the co-ordinates of the body at any time t of the motion.

The former gives $t = \frac{x}{u \cos \alpha}$, which substituted in the latter gives

$$y = \frac{x \sin \alpha}{\cos \alpha} - \frac{1}{2} \cdot \frac{g x^2}{u^2 \cos^2 \alpha} \\ = x \tan \alpha - \frac{g \sec^2 \alpha}{2 u^2} \cdot x^2 \dots \dots (3),$$

which is the equation of the path of the body.

77. It can readily be shewn that this equation is that of the common parabola.

For, arranging it according to powers of x ,

$$x^2 - \frac{2 u^2}{g} \cdot \sin \alpha \cdot \cos \alpha \cdot x = - \frac{2 u^2}{g} \cdot \cos^2 \alpha \cdot y,$$

and completing the square,

$$\left(x - \frac{u^2}{g} \cdot \sin \alpha \cos \alpha\right)^2 = \frac{2 u^2}{g} \cdot \cos^2 \alpha \cdot \left(\frac{u^2}{2 g} \sin^2 \alpha - y\right),$$

and by transposing the origin of co-ordinates from A to a certain point, the co-ordinates of which are

$$\frac{u^2}{g} \cdot \sin \alpha \cos \alpha, \quad \frac{u^2}{2 g} \sin^2 \alpha ;$$

and reckoning the new ordinate axis of y downwards, which will be effected by writing

$$x' + \frac{u^2}{g} \sin \alpha \cos \alpha \text{ for } x,$$

$$\text{and } -y' + \frac{u^2}{2 g} \cdot \sin^2 \alpha \text{ for } y,$$

we find

$$x'^2 = \frac{2 u^2}{g} \cos^2 \alpha \cdot y',$$

which is the equation of a parabola of which the vertex is the origin, and axis downwards.

$$\text{Its latus rectum} = \frac{2 u^2}{g} \cdot \cos^2 \alpha.$$

78. To find the greatest altitude the projectile attains, and its horizontal range, we have

$$\left(x - \frac{u^2}{g} \cos \alpha \sin \alpha\right)^2 = \frac{2u^2}{g} \cdot \cos^2 \alpha \cdot \left(\frac{u^2}{2g} \sin^2 \alpha - y\right).$$

Now because the left hand side of this equation is an exact square, it cannot be equal to a negative quantity;

$$\therefore y \text{ cannot be } > \frac{u^2}{2g} \cdot \sin^2 \alpha.$$

Hence the *greatest vertical altitude* the body attains above the point of projection, being the greatest value of y , is

$$= \frac{u^2}{2g} \cdot \sin^2 \alpha.$$

Again. If the parabola intersect the horizontal line Ax , passing through the point of projection A , in B , AB is called the horizontal range, and its value may be found by considering that at B , $y = 0$ and $x = AB$, which being substituted in equation (3) of Art. 76, give

$$0 = AB \cdot \tan \alpha - \frac{g \sec^2 \alpha}{2u^2} \cdot AB^2;$$

$$\therefore AB = \frac{u^2}{g} \cdot \sin 2\alpha, \text{ is the horizontal range.}$$

79. We shall shew how these results may be directly obtained by the second Law of Motion, without the aid of the equation of the path of the projectile.

1st. To find the greatest altitude, time of flight, and range.

Let the velocity of projection be resolved into components, one $= u \cos \alpha$ in a horizontal direction, and the other $= u \sin \alpha$ in a vertical direction. The former of these does not affect the vertical motion; as far therefore as the upwards motion of the body is concerned, we may consider it to have been projected vertically with the velocity $u \sin \alpha$: and it will

continue to ascend till all this is destroyed by the force of gravity. Hence, because in general $s = \frac{v^2}{2f}$ (Art. 60), in this case

$$\text{the altitude attained} = \frac{(u \sin \alpha)^2}{2g};$$

and since all the vertical velocity is lost in ascending,

$$\therefore \text{time of ascent} = \frac{u \sin \alpha}{g} \text{ (Art. 57).}$$

But the time of descent is equal to the time of ascent,

$$\therefore \text{time of flight} = \frac{2u \sin \alpha}{g}.$$

At the expiration of this time the body strikes the horizontal plane, having this while been moving horizontally with the *uniform* velocity $u \cos \alpha$; consequently the horizontal range, by Art. 5,

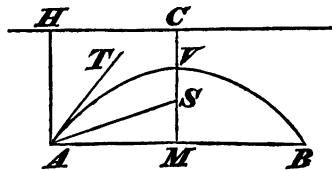
$$= (\text{vel.}) \cdot (\text{time}) = u \cos \alpha \cdot \frac{2u \sin \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}.$$

The velocity of the projectile in a horizontal direction is uniform by the first Law of Motion; for gravity, the only force, acting in a vertical direction, there is no horizontal force acting upon the body.

Remark. The range will be the greatest for a given velocity of projection, when $\sin 2\alpha$ is the greatest, *i.e.* when $\alpha = 45^\circ$.

80. *To find the positions of the focus and directrix.*

Let the horizontal range AB be bisected in M by the vertical line VM ; this line is the axis of the parabola. From A the point of projection draw AH vertical to meet the directrix in H ; and AS making the $\angle SAT = \angle HAT$, AT being the direction of projection. By



the property of the parabola, AT being a tangent at A , bisects the angle between AH and the radius vector drawn from A to the focus. Hence the focus is in the line AS ; it is also in VM ; and therefore S is the focus, and $AH = AS$.

$$\text{Now } AM = \frac{1}{2} \text{ the range } AB = \frac{u^2 \sin 2\alpha}{2g}.$$

$$\text{And } \angle SAH = 2 \angle TAH = 2 \left(\frac{\pi}{2} - \alpha \right) = \pi - 2\alpha,$$

$$\therefore \angle SAM = \frac{\pi}{2} - \angle SAH = \frac{\pi}{2} - (\pi - 2\alpha) = 2\alpha - \frac{\pi}{2} \dots\dots(1);$$

$$\begin{aligned} \therefore HA = AS &= \frac{AM}{\cos SAM} \\ &= \frac{u^2 \sin 2\alpha}{2g \sin 2\alpha} = \frac{u^2}{2g} \dots\dots\dots(2). \end{aligned}$$

Equation (1) determines the position of AS in which the focus lies; and (2) determines the distance of both focus and directrix from the point of projection.

Remark. When the range is the greatest $\sin 2\alpha = 1$, or $2\alpha - \frac{\pi}{2} = 0$, or $SAM = 0$, and therefore the focus of the parabola is in the horizontal plane AB .

81. *The velocity at any point of the parabolic path is equal to that which a body would acquire in falling from rest vertically from the directrix to that point.*

For any point of the path may be considered a point of projection, the tangent at that point being the direction of motion at that instant. Hence A may be taken to represent the place of the body at any moment, and AT the direction of its motion. Now by equation (2) of last article $u^2 = 2g \cdot HA$, which, being compared with the formula $v^2 = 2fs$ for falling bodies in Art. 60, shews that u is the velocity which a body would acquire in falling from rest from H to A .

Remark. From this property we can find the length of the latus rectum. For, the velocity at V being wholly horizontal $= u \cos \alpha$; and consequently the space due to this

velocity, or distance of V below the directrix (which = $\frac{1}{2}$ lat. rect.) = $\frac{(u \cos \alpha)^2}{2g}$;

$$\therefore \text{lat. rect.} = \frac{2u^2 \cos^2 \alpha}{g}.$$

82. To find the length of the whole path APB.

Let $s = AP$, and write p for $d_x y$.

$$\text{Then } s = \int_x \sqrt{1 + p^2} = \int_p \sqrt{1 + p^2} \cdot d_p x.$$

$$\text{Now } y = x \tan \alpha - \frac{g \sec^2 \alpha}{2u^2} \cdot x^2;$$

$$\therefore p = \tan \alpha - \frac{g \sec^2 \alpha}{u^2} \cdot x \dots \dots (1);$$

$$\therefore d_x p = -\frac{g \sec^2 \alpha}{u^2};$$

$$\therefore d_p x = \frac{1}{d_x p} = -\frac{u^2}{g \sec^2 \alpha} = -\frac{u^2}{g} \cdot \cos^2 \alpha;$$

$$\begin{aligned} \therefore s &= -\int_p \sqrt{1 + p^2} \cdot \frac{u^2}{g} \cos^2 \alpha = -\frac{u^2}{g} \cos^2 \alpha \int_p \sqrt{1 + p^2} \\ &= -\frac{u^2}{2g} \cos^2 \alpha \{p \sqrt{1 + p^2} + \log_e (p + \sqrt{1 + p^2}) + C\}. \end{aligned}$$

The value of p at the point of projection A is $\tan \alpha$ from (1), and at B it is $-\tan \alpha$. Hence the above integral is to be taken between $p = \tan \alpha$ and $p = -\tan \alpha$;

$$\begin{aligned} \therefore \text{arc } APB &= \frac{u^2}{2g} \cos^2 \alpha \{2 \tan \alpha \cdot \sec \alpha + \log_e (\sec \alpha + \tan \alpha) \\ &\quad - \log_e (\sec \alpha - \tan \alpha)\} \\ &= \frac{u^2}{g} \left\{ \sin \alpha + \cos^2 \alpha \log_e \tan \left(45^\circ + \frac{\alpha}{2} \right) \right\}. \end{aligned}$$

PROBLEMS.

1. *Two bodies being projected at the same moment in a vertical direction with given velocities from two points in a horizontal plane; it is required to find the greatest altitude to which their common centre of gravity will rise.*

Let A, B be the masses of the bodies,

a, b their velocities of projection upwards,

then the velocity with which their common centre of gravity will begin to ascend

$$= \frac{Aa + Bb}{A + B} \dots\dots\dots \text{Art. 49.}$$

By Art. 53 the motion of the centre of gravity will be the same as that of a body $A + B$, projected upwards with this velocity; and therefore the greatest height to which it will ascend equals the space through which a body must fall from rest to acquire this velocity (Art. 70)

$$= \frac{(\text{vel.})^2}{2g} \dots\dots\dots \text{Art. 60,}$$

$$= \frac{(Aa + Bb)^2}{2g(A + B)^2}.$$

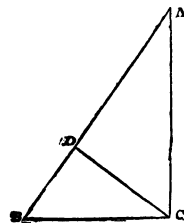
2. *Two bodies start from the top of an inclined plane, one falling down the plane, and the other down the vertical altitude, and it is observed that the former is twice as long as the latter in reaching the base. What is the inclination of the plane to the horizon?*

Let AB be the inclined plane, AC its vertical altitude.

Then g being the force of gravity, to which the motion of the body which descends from A to C is due, its time of motion

$$= \left(\frac{2 \cdot AC}{g} \right)^{\frac{1}{2}} \dots\dots\dots \text{Art. 60.}$$

and $g \sin B$ being the force down the plane



(Art. 65) to which the motion of the other is due, its time of motion

$$= \left(\frac{2AB}{g \sin B} \right)^{\frac{1}{2}};$$

and by the question this is double the former;

$$\therefore \left(\frac{2AB}{g \sin B} \right)^{\frac{1}{2}} = 2 \left(\frac{2AC}{g} \right)^{\frac{1}{2}};$$

$$\therefore \frac{AB}{AC} \cdot \sin B = \frac{1}{4},$$

$$\text{or } (\sin B)^2 = \frac{1}{4};$$

$$\therefore \sin B = \frac{1}{2} = \sin 30^\circ;$$

$$\therefore B = 30^\circ.$$

3. *Two bodies are connected by a very fine string which passes over the top of a given inclined plane. One of them hanging down draws the other up the plane. Required the whole time of motion.*

Let AB be the inclined plane.

P, Q the masses of the bodies.

Then, by Statics, the force acting up the plane which would just prevent Q from descending

$$= Qg \sin B.$$

This then is the pressure of Q , which is wholly employed in opposing P 's descent. The pressure which P exerts to descend

$$= Pg.$$

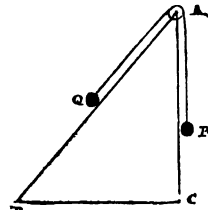
Consequently, the whole pressure which produces the motion of the system

$$= Pg - Qg \sin B.$$

The mass moved $= P + Q$;

therefore the accelerating force on the system

$$= \frac{P - Q \sin B}{P + Q} \cdot g \dots\dots\dots \text{Art. 23,}$$



which being constant, if t be the time spent in ascending from B to A ,

$$AB = \frac{1}{2} \cdot \frac{P - Q \sin B}{P + Q} \cdot g t^2 \dots\dots\dots \text{Art. 60,}$$

from which equation t is known.

4. *A given body Q is to be raised to a given altitude by means of a smaller body P; required the length of the plane upon which this may be effected in the least possible time.*

Let $AB = x$ be the required plane (Fig. of last Art.),

$AC = a$ the given altitude.

Then as in the last problem,

$$x = \frac{1}{2} \cdot \frac{P - Q \cdot \frac{a}{x}}{P + Q} \cdot g t^2,$$

$$\therefore t^2 \propto \frac{x}{P - Q \cdot \frac{a}{x}},$$

and since t is to be a minimum, t^2 must be a minimum, and its reciprocal which

$$\propto \frac{P}{x} - \frac{Qa}{x^2},$$

must be a maximum. Putting the differential coefficient of this expression = 0,

$$0 = P - \frac{2Qa}{x},$$

$$\text{or } x = \frac{Q}{P} \cdot 2a,$$

which is twice the length necessary for statical equilibrium.

5. *A body is projected vertically upwards with a given velocity; required the time of its being at a given altitude above the point of projection.*

Let a = the given altitude,

u = the given velocity,

t = the required time.

Then the space which would be described in consequence of the projectile velocity, in the time t ,

$$= ut,$$

and the space which would be fallen through in consequence of the action of gravity in the same time

$$= \frac{1}{2}gt^2,$$

and a is the space which results from these two motions (second Law of Motion);

$$\therefore a = ut - \frac{1}{2}gt^2 \dots \text{as in Art. 62};$$

$$\therefore t = \frac{u}{g} \pm \frac{(u^2 - 2ag)^{\frac{1}{2}}}{g} \dots \dots (1).$$

If $u^2 < 2ag$ the question is impossible, therefore the body cannot ascend higher than $\frac{u^2}{2g}$. Let A be the point of projection, AB the given altitude; $AC = \frac{u^2}{2g}$ = the greatest altitude to which it ascends. The body after leaving A passes through B for the first time, after which it continues its ascent to C , when it begins to fall towards A , and in its descent passes through B a second time. The two values of t in equation (1) refer to this circumstance. Hence the body is at the given altitude at the time

$$\frac{u}{g} - \frac{(u^2 - 2ag)^{\frac{1}{2}}}{g} \text{ in its ascent};$$

and at the time

$$\frac{u}{g} + \frac{(u^2 - 2ag)^{\frac{1}{2}}}{g} \text{ in its descent.}$$

6. *A body falling from the top of a tower was observed to descend through $\left(\frac{1}{n}\right)^{\text{th}}$ part of its altitude in the last second. Required the whole time of descent, and the altitude of the tower.*

Let t = whole time of descent;

$$\therefore \text{the altitude of the tower} = \frac{1}{2}gt^2,$$

and the space descended in the time $(t-1) = \frac{1}{2}g(t-1)^2$;

$$\therefore \text{the space descended in the last second} = \frac{1}{2}g(2t-1),$$

$$\text{and } \frac{1}{2}gt^2 = n \cdot \frac{1}{2}g(2t-1) \text{ by the question;}$$

$$\therefore t^2 = 2nt - n;$$

$$\therefore t = n \pm \sqrt{n^2 - n}.$$

The double sign in this equation seems to imply that the question is ambiguous: but since $t-1$ must be a positive quantity, and $n > 1$ by the nature of the question, the lower sign cannot be used. For, if we take the lower sign,

$$t-1 = n-1 - \sqrt{n^2 - n}$$

$$= \frac{(n-1)^2 - (n^2 - n)}{n-1 + \sqrt{n^2 - n}}$$

$$= - \frac{n-1}{n-1 + \sqrt{n^2 - n}} \text{ a negative quantity;}$$

$$\therefore t = n + \sqrt{n^2 - n},$$

and the altitude of the tower $= \frac{1}{2}gn(2n-1 + 2\sqrt{n^2 - n})$.

7. *An elastic body falls from a given altitude upon a horizontal plane; and rebounds and descends again, and so on till the motion ceases. Required the whole space described.*

Let a = the given altitude,

λ = the elasticity of the body.

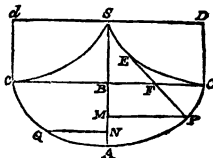
Then the velocity acquired in 1st descent $= \sqrt{2ga}$;

$$\therefore \text{velocity of 1st rebound} = \lambda \sqrt{2ga};$$

$$\begin{aligned}
 \therefore \text{space described in 1st ascent} &= \frac{\lambda^2 \cdot 2ga}{2g} \\
 &= \lambda^2 a = \text{space described in 2nd} \\
 &\quad \text{descent;} \\
 \therefore \text{space described in 2nd ascent} &= \lambda^2 (\lambda^2 a) \\
 &= \lambda^4 a = \text{space described in 3rd} \\
 &\quad \text{descent.} \\
 \&c. &= \&c.; \\
 \therefore \text{whole space} &= a + 2\lambda^2 a + 2\lambda^4 a + 2\lambda^6 a + \dots \text{ad infinitum} \\
 &= a + 2\lambda^2 a (1 + \lambda^2 + \lambda^4 + \dots) \\
 &= a + \frac{2\lambda^2 a}{1 - \lambda^2} \\
 &= \frac{1 + \lambda^2}{1 - \lambda^2} \cdot a.
 \end{aligned}$$

8. *Two elastic bodies oscillating in the same cycloid, through unequal arcs, strike together at the lowest point, and rebounding through other arcs descend and strike again; this is repeated indefinitely. To find the velocity of each after any number of impacts.*

Let m, m' be the masses of the bodies, the former starting from P and the latter from Q . They strike together at A by the question: and since the cycloid is an isochronous curve (Art. 78), whatever may be the velocities with which they separate, they will ascend and descend through their respective arcs in equal times, and meet again at A . Let u, u' be their velocities at first impact; and λ their elasticity: and v, v' their velocities after the $(n)^{\text{th}}$ impact.



Then their relative velocity at 1st impact $= u + u'$.
Hence the relative velocity with which they separate, which is
 $=$ their relative velocity at 2nd impact $= \lambda (u + u')$ Art. 36.
Similarly 3rd $= \lambda^2 (u + u')$,
..... $(n)^{\text{th}}$ $= \lambda^{n-1} (u + u')$;

$\therefore$ their relative velocity after the $(n)^{\text{th}}$ impact $= \lambda^n(u + u')$,
this is equal to $v + v'$.

Now the momentum gained by one is lost by the other at every impact and therefore the algebraic sum of the momenta is always the same (Art. 27) ;

$$\therefore mu - m'u' = mv - m'v',$$

$$\text{but } \lambda^n(u + u') = v + v';$$

$$\therefore v = \frac{mu - m'u'}{m + m'} + \lambda^n \cdot \frac{m'(u + u')}{m' + m},$$

$$\text{and } v' = \frac{mu - m'u'}{m' + m} + \lambda^n \cdot \frac{m(u + u')}{m + m'}.$$

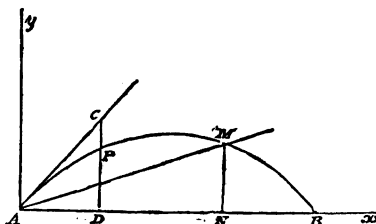
9. *A body is projected from a given point in a given direction with a given velocity ; to find the time of flight and its range upon an inclined plane passing through the point of projection.*

Let AC be the direction of projection, AM the inclined plane passing through A the point of projection: draw Ay , Ax vertical and horizontal, and let APM be the path of the body.

u = velocity of projection,

$\alpha = BAC$, $i = BAM$,

$x = AN$, $y = MN$.



Then because xy are the co-ordinates of a point M in the inclined plane,

$$\therefore y = x \tan i.$$

But because they are also co-ordinates of a point M in the path of a projectile,

$$\therefore y = x \tan \alpha - \frac{g \sec^2 \alpha}{2u^2} \cdot x^2 \dots \dots \text{Art. 76, (3)};$$

$$\therefore x \tan i = x \tan \alpha - \frac{g \sec^2 \alpha}{2u^2} \cdot x^2.$$

Hence dividing by x , we find,

$$\begin{aligned} x &= \frac{2u^2}{g} \cdot \cos^2 \alpha (\tan \alpha - \tan i) \\ &= \frac{2u^2}{g} \cdot \frac{\cos \alpha \cdot \sin (\alpha - i)}{\cos i}; \end{aligned}$$

$$\therefore \text{the range } AM = \frac{x}{\cos i} = \frac{2u^2}{g} \cdot \frac{\cos \alpha \cdot \sin (\alpha - i)}{\cos^2 i} \dots\dots (1).$$

Again, since the horizontal velocity is uniform and equal to $u \cos \alpha$,

$$\text{the time of flight} = \frac{x}{u \cos \alpha} \dots\dots\dots \text{Art. 76, (1)}$$

$$= \frac{2u}{g} \cdot \frac{\sin (\alpha - i)}{\cos i} \dots\dots\dots (2).$$

10. *To find at what angle of elevation the body in the last question must be projected with a given velocity, that the range on the inclined plane may be the greatest possible.*

Let θ = the elevation above the horizon = CAX .

Then, as in the last question,

$$\text{the range } AM = \frac{2u^2}{g} \cdot \frac{\cos \theta \cdot \sin (\theta - i)}{\cos^2 i}.$$

This quantity is to be made a maximum by the variation of θ : hence omitting constant factors we have to find the value of θ , which makes

$$2 \cos \theta \cdot \sin (\theta - i) \text{ a maximum,}$$

$$\text{or } \sin (2\theta - i) - \sin i \text{ a maximum.}$$

This will be the case when $\sin (2\theta - i) = 1$,

$$\text{or } 2\theta - i = 90^\circ;$$

$$\therefore \theta = 45^\circ + \frac{i}{2}.$$

Remark. Since $2\theta - i = 90^\circ$; $\therefore \theta - i = 90^\circ - \theta$,

that is, $CAx - MAx = yAx - CAx$,

or $CAM = CAy$.

Hence the direction of projection must bisect the angle between the plane and the vertical; and therefore the range is the greatest when the focus of the parabola is in the inclined plane.

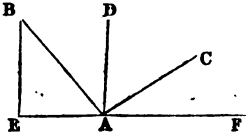
11. *An elastic body slides down an inclined plane of given length, and impinges on the horizontal plane at the foot of the given plane: required the inclination of the given plane that the range of the body after reflection at the horizontal plane may be a maximum.*

Let a = the length of the plane AB ;

$\theta = BAE$ its inclination,

$\theta' = CAF$, AC being the direction in which the body is reflected at A ,

λ = the elasticity.



Draw AD , BE vertical. Then the velocity which the body acquires in falling from B to A ,

$$= \sqrt{2g \cdot BE} = \sqrt{2ga \sin \theta} \dots \dots \dots \text{Art. 66.}$$

This being the velocity with which it impinges against the plane AF at the angle of incidence $DAB = 90^\circ - \theta$; and $90^\circ - \theta' = DAC$ being the angle of reflection, by hypothesis;

$$\begin{aligned} \therefore \text{the vel. of reflection} &= \frac{\sin(90^\circ - \theta)}{\sin(90^\circ - \theta')} \cdot \sqrt{2ga \sin \theta} \dots \text{Art. 44.} \\ &= \frac{\cos \theta}{\cos \theta'} \cdot \sqrt{2ga \sin \theta}. \end{aligned}$$

To find the range upon the horizontal plane AF , we must consider the body projected from A with this velocity in the direction AC , such that $CAF = \theta'$;

$$\therefore \text{the range} = \frac{\cos^2 \theta}{\cos^2 \theta'} \cdot \frac{2ga \sin \theta}{g} \cdot \sin 2\theta' \dots\dots\dots \text{Art. 78.}$$

$$= 4a \cos^2 \theta \sin \theta \cdot \tan \theta'.$$

$$\text{But } \tan (90^\circ - \theta') = \frac{1}{\lambda} \cdot \tan (90^\circ - \theta),$$

$$\text{or } \tan \theta' = \lambda \cdot \tan \theta;$$

$$\therefore \text{the range} = 4a\lambda \cos^2 \theta \sin \theta \tan \theta \\ = 4a\lambda (\cos \theta - \cos^3 \theta).$$

This is to be made a maximum by the variation of θ ; hence according to the principles of the Differential Calculus,

$$0 = -4a\lambda (\sin \theta - 3 \cos^2 \theta \cdot \sin \theta);$$

$$\therefore \theta = \sec^{-1} \sqrt{3}.$$

12. *From the top of a tower two bodies are projected with the same given velocity at given angles of elevation, and are observed to strike the ground at the same point. Required the altitude of the tower.*

Take the top of the tower for the origin of co-ordinates, the axis of x being horizontal and that of y vertical.

x = distance of the point struck from the foot of the tower,

y = altitude of the tower,

θ = any angle of elevation of projection, such that the body may strike the ground at the point mentioned in the question,

u = the velocity of projection.

Then, the co-ordinates of the point on the ground against which the balls strike are x and $-y$; and because this is a point in the path of a projectile,

$$\therefore -y = x \tan \theta - \frac{g \sec^2 \theta}{2u^2} \cdot x^2 \dots\dots\dots \text{Art. 76, (3)}$$

$$= x \tan \theta - \frac{g x^2}{2u^2} \cdot (1 + \tan^2 \theta);$$

$$\therefore \tan^2 \theta - \frac{2u^2}{g\alpha} \tan \theta + 1 - \frac{2u^2 y}{g\alpha^2} = 0.$$

This being a quadratic equation has two roots, or values of $\tan \theta$, and no more: hence there are two directions, and no more, in which bodies may be projected with a given velocity to strike the same point; these being given by the question call them α and β , then $\tan \alpha$ and $\tan \beta$ are the two roots of the preceding quadratic; wherefore by the nature of equations,

$$\tan \alpha + \tan \beta = \frac{2u^2}{g\alpha},$$

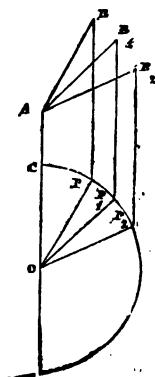
$$\text{and } \tan \alpha \tan \beta = 1 - \frac{2u^2 y}{g\alpha^2}$$

$$= 1 - \frac{gy}{2u^2} (\tan \alpha + \tan \beta)^2;$$

$$\begin{aligned} \therefore y &= \frac{2u^2}{g} \cdot \frac{1 - \tan \alpha \tan \beta}{(\tan \alpha + \tan \beta)^2} \\ &= \frac{2u^2}{g} \cdot \frac{\cos \alpha \cos \beta \cos (\alpha + \beta)}{\sin^2 (\alpha + \beta)}. \end{aligned}$$

13. *An infinite number of bodies are projected from a given point, with the same velocity, in all directions; to find the locus of them all at a given time.*

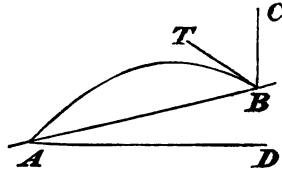
Let $AB, AB_1, AB_2, \dots$ be the spaces that would be described in any time with the velocity and directions of projection of the bodies $P, P_1, P_2, \dots$; these spaces are all equal, because the velocity of projection is the same for all; draw $PB, P_1B_1, P_2B_2, \dots$ vertical; these also are all equal, because they are the spaces due to the action of gravity on the bodies during the time elapsed from projection. Complete the parallelograms $BO, B_1O, B_2O, \dots$ therefore $OP = OP_1 = OP_2 = \dots$; hence the bodies are situated in the surface of a sphere, whose centre is O .



The following problem is added as an application of the principle of the second Law of Motion, that forces acting simultaneously upon a body produce their effects independently.

14. *A perfectly elastic ball is thrown from a point A of a given inclined plane, and after impinging on the plane at B rises vertically through a given altitude BC; after which it again impinges at B and returns to A. Find the range AB and time of flight.*

Let BC , which is given, $= h$, and put $\angle BAD$, the inclination of the plane, $= i$. Since the velocity is not affected by impact at B , the body in returning from C will exactly retrace its former path from A to C ; and (by Art. 44, Remark) if BT be the direction of reflection at B , BT , BC make equal angles with the normal to the plane at B ,



$$\therefore TBA = \frac{\pi}{2} - i.$$

Now the velocity of impact at $B = \sqrt{2gh}$ = vel. of projection in the direction BT : let both this velocity, and the force of gravity be resolved perpendicular and parallel to the plane; these are

$$\left\{ \begin{array}{l} \text{vel.} = \sqrt{2gh} \cdot \sin i \\ \text{force} = g \cdot \sin i \end{array} \right\} \begin{array}{l} \text{parallel} \\ \text{to plane.} \end{array} \quad \text{and} \quad \left\{ \begin{array}{l} \text{vel.} = \sqrt{2gh} \cdot \cos i \\ \text{force} = g \cdot \cos i \end{array} \right\} \begin{array}{l} \text{perpend.} \\ \text{to plane.} \end{array}$$

By the second Law of Motion we may consider these sets of velocities and forces separately. Taking the latter set, the body is projected perpendicular to the plane with the velocity $\sqrt{2gh} \cos i$, subject to the retarding force $g \cos i$; this velocity will be destroyed in the time

$$\frac{\sqrt{2gh} \cdot \sin i}{g \sin i} = \left(\frac{2h}{g} \right)^{\frac{1}{2}} \dots \text{Art. 57,}$$

which equals the time of descending again to the plane.

Hence t the time of flight $= 2 \left(\frac{2h}{g} \right)^{\frac{1}{2}}$,

which we may remark is independent of the plane's inclination.

Taking the other set, the body is projected down the plane with the velocity $\sqrt{2gh} \sin i$, and is accelerated by the force $g \cos i$, and arrives at A at the time t ; hence by Art. 62,

$$AB = t \sqrt{2gh} \sin i + \frac{1}{2} g \sin i \cdot t^2 = 8h \sin i.$$

CHAPTER IV.

ON THE FREE MOTION OF A PARTICLE OF MATTER WHEN ACTED ON BY VARIABLE FORCES.

83. *A PARTICLE of matter in motion being acted on by a force in the line of motion, it is required to investigate the differential equations expressing the relations which exist between force, the space described, the velocity and the time of motion.*

Let s be the distance of the particle from some fixed point in the line of motion, at the time t reckoned from some fixed epoch; v the velocity and f the accelerating force at the same time. Also let $s \pm \delta s$, $f + \delta f$, $v + \delta v$ be the values of s , f and v at the time $t + \delta t$. (In the expression $s \pm \delta s$ the upper or lower sign is to be used according as the motion of the particle is *from* or *towards* the point from which s is measured; in other words, according as s *increases* or *decreases* with t .) Now since the space $\pm \delta s$ is described in the time δt with velocity varying between v and $v + \delta v$,

$$\therefore \pm \delta s \text{ lies between } v\delta t \text{ and } (v + \delta v)\delta t,$$

$$\text{or } \pm \frac{\delta s}{\delta t} \dots\dots\dots v \dots\dots v + \delta v.$$

As this is the case however small δt be taken, it will be true when δt is diminished indefinitely; in which case $\frac{\delta s}{\delta t}$ approximates to $d_t s$ as its limit, by the principles of the *Diff. Cal.*, and v and $v + \delta v$ approximate to equality with each other;

$$\therefore \pm d_t s = v \dots\dots (1).$$

84. Again, δv is the velocity added in the time δt by a force varying in magnitude from f to $f + \delta f$,

$$\therefore \delta v \text{ lies between } f\delta t \text{ and } (f + \delta f)\delta t,$$

$$\text{or } \frac{\delta v}{\delta t} \dots\dots\dots f \dots\dots\dots f + \delta f.$$

Hence as before $d_t v = f$.

85. If f be a retarding force, it may be shewn in the same way that $-d_t v = f$. It appears, therefore, that in general

$$f = \pm d_t v, \dots\dots (2),$$

the sign + or - being used, according as v *increases* or *decreases*.

86. Combining equations (1) and (2) together, we have

$$f = \pm d_t (\pm d_t s) = \pm \pm d_t^2 s.$$

Now, if f be an *accelerating* force, and s *increase* with t , this becomes $f = + + d_t^2 s = d_t^2 s$. And if f be a *retarding* force and s *decrease*, this becomes $f = - - d_t^2 s = d_t^2 s$. In the former of these cases the tendency of f is to *increase* s : and in the latter f retards the decrement of s , which is only another way of saying that f acts so as to increase s : consequently, *when the force f acts to increase s , the equation for f in terms of s is*

$$f = + d_t^2 s.$$

Again, if f be an *accelerating* force, and s *decrease*, $f = + - d_t^2 s = - d_t^2 s$; and if f be a *retarding* force, and s *increase*, $f = - + d_t^2 s = - d_t^2 s$. But in the former of these cases the tendency of f is to increase the decrement of s ; that is, it acts to diminish s : and in the latter case the tendency of f is to diminish the increment of s ; that is, it acts to diminish s ; consequently, *when the force f acts to diminish s , the equation for f in terms of s is $f = - d_t^2 s$.*

We, therefore, state in general that

$$f = \pm d_t^2 s \dots\dots (3),$$

the upper or lower sign being used according as f acts to increase or diminish s .

87. There is another expression for f in terms of v and s , which may be thus deduced.

$$\begin{aligned}
 \text{From equation (2) we have } f &= \pm d_t v \\
 &= \pm d_t s \cdot d_t v \\
 &= \pm \pm v d_t v \text{ from (1).}
 \end{aligned}$$

Here the union of signs is precisely the same as in the last article, and therefore, by proceeding in the same way here, we obtain the equation

$$f = \pm v d_t v \dots\dots (4),$$

the upper or lower sign being used according as f acts to increase or diminish s .

We shall subjoin a few examples of the application of these formulæ.

In the last chapter the force of gravity was stated to be uniform at the same place. This is found to be true, and then only approximately, for small altitudes above the Earth's surface. At the top of a mountain the force is sensibly less than in the plains below. In the Treatise on Attractions appended to this book, it will be shewn that the Earth's attractive force on external objects varies as the square of their respective distances from its centre inversely; but on internal objects as the distance from the centre directly. Hence the results of Chapter III. are only to be applied when bodies ascend or descend through spaces which are very small in comparison of the Earth's radius, which is a little less than 4000 miles. We shall proceed to solve the problem in a more general form.

88. *PROB. A body falls from a great altitude towards the Earth, to determine the motion. In this case the force varies inversely as the square of the distance from the Earth's centre.*

Let S be the Earth's centre, A the point from which the body begins to fall, P the place of the body, at any time t from the beginning of the motion. v = its velocity, and $s = SP$, $a = SA$, R = the Earth's radius, and g = the force of gravity at the Earth's surface.

A —
P —
S —

Then g : accelerating force at $P :: \frac{1}{R^2} : \frac{1}{s^2}$;

$$\begin{aligned}\therefore \text{accelerating force at } P &= \frac{R^2 g}{s^2} \\ &= \frac{\mu}{s^2}, \text{ if we write } \mu \text{ for } R^2 g.\end{aligned}$$

Now this force acts to diminish s ;

$$\therefore \frac{\mu}{s^2} = -v d_s v \dots\dots \text{Art. 87.}$$

Multiply by -2 , and integrate;

$$\therefore \frac{2\mu}{s} + \text{constant} = v^2.$$

Now $v = 0$, when $s = a$; $\therefore \text{constant} = -\frac{2\mu}{a}$;

$$\therefore v^2 = 2\mu \left(\frac{1}{s} - \frac{1}{a} \right) \dots\dots (1);$$

$$\therefore -d_s s = v = \sqrt{2\mu} \cdot \left(\frac{1}{s} - \frac{1}{a} \right)^{\frac{1}{2}};$$

the sign $-$ is prefixed to $d_s s$, because s decreases as t increases (Art. 88).

$$\begin{aligned}\therefore d_s t &= - \left(\frac{a}{2\mu} \right)^{\frac{1}{2}} \cdot \frac{s}{\sqrt{as - s^2}} \\ &= \left(\frac{a}{2\mu} \right)^{\frac{1}{2}} \cdot \left\{ \frac{\frac{1}{2}a - s}{\sqrt{as - s^2}} - \frac{a}{2} \cdot \frac{1}{\sqrt{as - s^2}} \right\}; \\ \therefore t &= \left(\frac{a}{2\mu} \right)^{\frac{1}{2}} \cdot \left\{ \sqrt{as - s^2} - \frac{a}{2} \text{vers}^{-1} \frac{2s}{a} + C \right\}.\end{aligned}$$

Now $t = 0$ when $s = a$, and $\therefore C = \frac{a}{2} \cdot \text{vers}^{-1} 2 = \frac{a\pi}{2}$;

$$\therefore t = \left(\frac{a}{2\mu} \right)^{\frac{1}{2}} \cdot \left\{ \sqrt{as - s^2} - \frac{a}{2} \cdot \text{vers}^{-1} \frac{2s}{a} + \frac{a\pi}{2} \right\} \dots\dots (2).$$

Equation (1) gives the velocity and (2) the time, when the body is at a point whose distance from the centre is s .

If it were possible for a body to fall to the very centre S subject to the action of the force which varies as the square of the distance inversely, the time of arriving there would

$$= \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \cdot \frac{a\pi}{2} = \frac{a^{\frac{3}{2}}\pi}{2\sqrt{2\mu}},$$

which is found by making $s = 0$, in the general expression for t .

We have shewn that the attractive force at $P = \frac{R^2 g}{s^2}$; now if P be such a point that $SP = 1$, or if SP be the unit of distance, the force at $P = R^2 g = \mu$. Hence the quantity μ is the attractive force at the unit of distance, and is generally called the *absolute* accelerating force.

89. If SA be infinite, then $\frac{1}{a} = 0$, and the velocity acquired by a body falling from an infinite distance to the point P

$$= \left(\frac{2\mu}{s}\right)^{\frac{1}{2}}.$$

90. PROB. *To investigate the motion of a particle descending from a point within the Earth towards its centre.* In this case the force varies as the distance.

Using the same notation and figure as in the last problem,

g : accelerating force at P :: R : s ;

$\therefore$ accelerating force at $P = \frac{g}{R} \cdot s$

$= \mu s$, writing μ for $\frac{g}{R}$ the absolute

force in this case;

$$\therefore \mu s = -v dv \dots \dots \dots \text{Art. 87.}$$

Multiply by -2 and integrate;

$$\therefore -\mu s^2 + \text{constant} = v^2.$$

But when $v = 0$, $s = a$, $\therefore$ constant $= \mu a^2$;

$$\therefore v^2 = \mu(a^2 - s^2) \dots \dots \dots (1);$$

$$\therefore -d_t s = v = \sqrt{\mu} \cdot \sqrt{a^2 - s^2} \dots \dots \dots (2);$$

— $d_t s$ is used for the same reason as in the last problem;

$$\therefore d_t t = -\frac{1}{\sqrt{\mu}} \cdot \frac{1}{\sqrt{a^2 - s^2}};$$

$$\therefore t = \frac{1}{\sqrt{\mu}} \cdot \cos^{-1} \frac{s}{a} + \text{constant}.$$

But when $t = 0$, $s = a$; $\therefore \text{constant} = 0$;

$$\therefore t = \frac{1}{\sqrt{\mu}} \cdot \cos^{-1} \frac{s}{a} \dots \dots \dots (3).$$

Equations (2) and (3) give the velocity and time when the body is at the point P , whose distance from the centre $= s$.

The time of arriving at the centre S will be found by writing 0 for s , and

$$\therefore = \frac{1}{\sqrt{\mu}} \cdot \cos^{-1} 0 = \frac{\pi}{2\sqrt{\mu}}.$$

It is remarkable that this is independent of the distance from the Earth's centre, of the point from which the body began to fall.

91. Remark. Equation (1) shews that the velocity is equal to zero, when

$$a^2 - s^2 = 0,$$

or when $s = a$, or $-a$

$$= SA, \text{ or } -SA.$$

This double value of s shews that the body after passing through the centre S proceeds to an equal distance, on the other side of it, as that from which it fell, before its velocity is lost; it then descends towards S passing through the centre again and stopping at A ; after which it falls again as before. Hence the time of a complete oscillation, is double that above found for the time of falling to the centre. The motion will go on for ever.

In the Prob. of Art. 88; the velocity is equal to zero, only when

$$s = a = SA;$$

it is infinite when

$$s = 0.$$

Also from the form of the equation it appears that s cannot be negative. In the equation for t the value of the arbitrary constant has been shewn to be

$$\frac{a}{2} \cdot \text{vers}^{-1} 2,$$

for which we have written $\frac{a\pi}{2}$. The general value however is $\frac{a(2r+1)\pi}{2}$, r being any positive or negative integer.

This seems to denote oscillation. Judging from the analytical results, it would seem that a body falling from *rest* at A descends to S where its velocity is infinite and changes sign; it then ascends to A , and thence descends as before. But there is a difficulty in perceiving how velocity can pass through an infinite value, so that the transition from $+\infty$ to $-\infty$ may be accomplished in a time so short that the body remains all the while *at*, or *within an insensible distance of*, the centre S . Newton in his Seventh Section has shewn, that if the body, at the time of being dropt from A , have an indefinitely small velocity communicated to it, in a direction at right angles to AS , it will describe an ellipse (whose foci are A, S) the minor axis of which is of insensible magnitude. If this hypothesis were allowable, the difficulty of interpreting the analytical results would be removed; inasmuch as the body would not then fall *into* the centre of force S , but would pass *round* it on the side opposite to A within an insensible distance, thus causing a change of sign in the velocity.

92. *To investigate differential equations for the motion of a free particle, acted on by any accelerating forces in the plane of motion; the particle and the forces being referred to fixed rectangular co-ordinates in that plane.*

Let x, y be the co-ordinates of the particle at the time t reckoned from some fixed epoch: and let X, Y be the sums of the accelerating forces, resolved respectively parallel to the axes of x, y . By the second Law of Motion, these produce their effects independently; and hence, considering X and Y as positive when they act in the directions of $+x$ and $+y$; and negative when they act in the directions of $-x$ and $-y$, we have from Arts. 83, 86,

$d_t x$ = velocity parallel to the axis of x ,

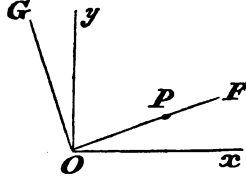
$d_t y$ = ... y ;

also $d_t^2 x = X$, and $d_t^2 y = Y$.

From the last two equations the motion is to be determined in every particular case by integration.

93. *To investigate the equations for the motion of a free particle, in one plane, when acted on by accelerating forces in that plane; the forces being referred to revolving co-ordinate axes, one of which always revolves with the particle.*

Let Ox, Oy be rectangular axes fixed in space; OF, OG revolving rectangular axes, one of which (OF) always passes through the particle P . Put $OP = r$, and $POx = \theta$; and let F, G be the forces acting on P , respectively parallel to OF, OG . Then the co-ordinates of P referred to the fixed axes Ox, Oy are $r \cos \theta, r \sin \theta$: and the forces acting on P in the same directions are $F \cos \theta - G \sin \theta, F \sin \theta + G \cos \theta$; and therefore by the last article,



$$d_t^2 (r \cos \theta) = F \cos \theta - G \sin \theta,$$

$$d_t^2 (r \sin \theta) = F \sin \theta + G \cos \theta;$$

or, performing the differentiations with regard to t ,

$$\{d_t^2 r - r(d_t \theta)^2\} \cos \theta - (2d_t r d_t \theta + r d_t^2 \theta) \sin \theta = F \cos \theta - G \sin \theta,$$

$$\{d_t^2 r - r(d_t \theta)^2\} \sin \theta + (2d_t r d_t \theta + r d_t^2 \theta) \cos \theta = F \sin \theta + G \cos \theta.$$

From which equations we obtain

$$d_t^2 r - r(d_t \theta)^2 = F \dots\dots\dots(1),$$

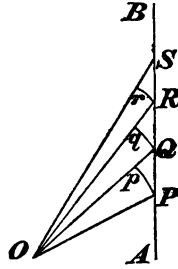
and $2d_t r d_t \theta + r d_t^2 \theta = G,$

or $\frac{1}{r} d_t(r^2 d_t \theta) = G \dots\dots\dots(2).$

94. Remark. If the particle P has no angular motion, OF is stationary, and θ is constant, and consequently $d_t^2 r = F$ in that case. By comparing this result with (1), it appears that the term $r(d_t \theta)^2$ is entirely owing to the circumstance of the angular motion of the line OF in which the particle is situated. It is important to remark this difference between the equations of motion of a particle, subject to the condition of being in a straight line which revolves about a fixed point, and those of a particle moving in a straight line which has no angular motion. We shall endeavour to explain this more fully.

95. From equation (1) we have $d_t^2 r = F + r(d_t \theta)^2$. Now $d_t^2 r$ is the *effective* force on the particle (Arts. 21, 86): that is, it is the force by which the motion of the particle has really been accelerated along the revolving line OF . Yet the impressed force in that direction is only F , which is less than the other by $r(d_t \theta)^2$. This seems to indicate that the particle is urged by some other force besides F ; namely, by a force equal to $r(d_t \theta)^2$. To impress this on the memory, it has been usual, in speaking of the motion of a particle which has angular motion about a fixed point, to say that it is acted on by a *centrifugal force* in addition to the actually impressed force. And $r(d_t \theta)^2$ being taken as the measure of this feigned force, it is evident that all equations formed upon the general results of Art. 92 for *fixed* axes will agree with equation (1) of Art. 93 for a *revolving* axis. It must not, however, be imagined that what is called centrifugal force is an actual force impressed upon the particle. The term $r(d_t \theta)^2$ which has obtained that appellation arises not from a force, but from the fact, stated in the first Law of Motion, that a particle's motion will be rectilinear and uniform unless it be compelled by external forces to move otherwise. To illustrate this, let

us consider the motion of a particle not acted on by a force. In this case the motion is uniform and rectilinear. Let AB be its path, and divide AB into very small equal parts $PQ, QR, RS, \dots$ these will represent the spaces described in very small equal times. Now although there is no acceleration of the actual velocity, yet if we measure the motion of the particle by its distances from any fixed point O , there is an acceleration of the velocity with which it recedes from O . For, join $OP, OQ, OR, OS, \dots$ and with centre O describe the small circular arcs $Pp, Qq, Rr, \dots$. Then $pQ, qR, rS, \dots$ are the spaces by which the particle has receded from O in the equal times of describing $PQ, QR, RS, \dots$. Now those spaces are greater as the body goes on, that is, $pQ < qR$, and $qR > rS$, &c. (for, ultimately, $Qp = PQ \cdot \cos OQA$, $Rq = QR \cos ORA$, &c.) Hence, measuring the motion from O , the particle instead of describing equal spaces describes *increasing* spaces, in equal times; its motion therefore when so measured seems to be due to an accelerating force tending from O (see Art. 5). This apparent force, though in reality no force at all, is that which for the reason above assigned has been called *centrifugal force*.



96. The student will see at once that if the distances of $P, Q, R, S \dots$ were measured from any *fixed line* whatever in space, the particle would describe equal distances (so measured) in equal times: and this is the reason why, when the motion is referred to fixed rectangular co-ordinates, there is no term corresponding to centrifugal force.

97. If while the line OF (fig. of Art. 93) revolves about O the particle P be kept at the same point of the line (by a string OP fastened at O , suppose), F will represent the force (the tension of the string) which restrains the particle; also, because in this case r is constant, $d_t^2 r = 0$; and therefore we have from equation (1)

$$F = -r(d_t \theta)^2.$$

The negative sign shews that the restraining force acts *from P towards O*; and, as the action of the particle must be exactly opposite to this, it is *from O towards P*, that is, *from the centre about which the motion of P is estimated*. It is on this account that $r (d_t \theta)^2$ is called *centrifugal force*.

98. If A denote the spiral area traced out in the time t by the line OP as radius vector, we know from the Differential Calculus that $d_t A = \frac{1}{2} r^2 d_t \theta$; and consequently making substitution in equation (2) of Art. 93 we have

$$d_t^2 A = 2 r G.$$

If therefore G be zero, and in no other case,

$$d_t^2 A = 0,$$

$$\therefore d_t A = C,$$

$$\text{and } A = Ct;$$

that is, *when the force acting on a particle is wholly central, and in no other case, the area swept out by the radius vector drawn from the particle to the centre of force, varies as the time.*

99. We have seen that $d_t s$ represents the *linear velocity*, or rate of increase of the line s ; for the same reason $d_t \theta$ represents the *angular velocity*, or rate of increase of the angle θ . Hence *the centrifugal force is equal to the radius vector multiplied into the square of its angular velocity.*

100. We have seen that $d_t^2 s$ represents the effective accelerating force to which the change of linear velocity is due; for a similar reason $d_t^2 \theta$ will represent the effective angular accelerating force, to which the change of angular velocity is due.

$d_t \theta$ and $d_t^2 \theta$ are subject to the same rules (Arts. 83, 86), with respect to their algebraic signs, as $d_t s$ and $d_t^2 s$.

101. Whenever the force impressed upon the particle is wholly central, or $G = 0$, equation (2) gives

$$d_t(r^2 d_t \theta) = 0,$$

$$\therefore r^2 d_t \theta = C,$$

$$\text{and } d_t \theta = \frac{C}{r^2}.$$

Wherefore, when a particle is acted on only by a central force, the angular velocity of its radius vector, about the centre, varies inversely as the square of its length.

And the centrifugal force, being equal to

$$r (d_t \theta)^2 = \frac{C}{r^3},$$

therefore varies, under the same condition, inversely as the cube of the radius vector.

102. In Article 93 we have estimated the impressed forces parallel and perpendicular to the radius vector of the particle acted on: *we shall now estimate them in the direction of a tangent and normal to the path of the particle.*

Let P be the place of the particle at the time t , APQ the path which it describes; PS , PR the tangent and normal at P ; S , R the forces acting in those directions. Ox , Oy the axes of co-ordinates; $x = OM$, $y = PM$, $S = AP$, ϕ the inclination of PS to Ox .

Resolving R and S parallel to Ox and Oy , we have

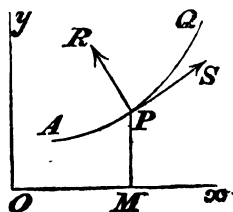
$$d_t^2 x = S \cos \phi - R \sin \phi,$$

$$d_t^2 y = S \sin \phi + R \cos \phi.$$

But because $\tan \phi = d_s y$,

$$\therefore \cos \phi = \frac{1}{\sqrt{1 + (d_s y)^2}} = d_s x = \frac{d_t x}{d_t s},$$

$$\text{and } \sin \phi = \frac{d_s y}{\sqrt{1 + (d_s y)^2}} = d_s y = \frac{d_t y}{d_t s};$$



wherefore by substituting these values,

$$d_t^3 x = S \frac{d_t x}{d_t s} - R \frac{d_t y}{d_t s},$$

$$d_t^3 y = S \frac{d_t y}{d_t s} + R \frac{d_t x}{d_t s}.$$

And, eliminating R , we obtain

$$\begin{aligned} d_t x d_t^2 x + d_t y d_t^2 y &= S \cdot \frac{(d_t x)^2 + (d_t y)^2}{d_t s} \\ &= S d_t s. \end{aligned}$$

But, because $(d_t x)^2 + (d_t y)^2 = (d_t s)^2$, by differentiating this equation we find,

$$\begin{aligned} d_t x d_t^3 x + d_t y d_t^3 y &= d_t s d_t^2 s; \\ \text{and } \therefore S &= d_t^2 s \dots\dots(1). \end{aligned}$$

This is the value of the *tangential* force.

Again, eliminating S , we obtain

$$d_t x d_t^2 y - d_t y d_t^2 x = R d_t s.$$

But if ρ be the radius of curvature of the path at P , we know from the Differential Calculus that

$$\rho = \frac{\{(d_t x)^2 + (d_t^2 y)^2\}^{\frac{3}{2}}}{d_t x d_t^2 y - d_t y d_t^2 x} = \frac{(d_t s)^3}{d_t x d_t^2 y - d_t y d_t^2 x};$$

ρ being measured here in the direction PR , or *from* the axis of x ; that is, it is *here* accounted *positive* when the curve is *convex* towards the axis of x , (which is the contrary to what is usually done in treatises on the Differential Calculus,) for the sake of accommodating our expressions to the figure, which must be convex to Ox if R act, as we have supposed it, from Ox .

$$\text{Hence } R = \frac{(d_t s)^2}{\rho} = \frac{(\text{velocity})^2}{\text{rad. curv.}} \dots\dots(2).$$

This is the value of the *normal* force.

103. The result expressed by equation (1) might have been predicted at once, by the aid of the second Law of Motion. For R , *always* acting at right angles to the element of the path at P , cannot accelerate the velocity with which that element is described: and therefore the acceleration of that velocity is wholly due to S , which is all that equation (1) expresses.

104. As we have above remarked, R acts *towards* the centre of curvature at P ; it is the force which curves the path. The reaction of the particle, or *force with which the particle resists the curving of its path*, is of course equal and opposite to R , and therefore *always tends from the centre of curvature, and is equal to the square of its velocity divided by the radius of curvature of its path at that point*. This is what is generally called *the centrifugal force* of the particle: and is to be carefully distinguished from that which in Art. 95 has been called by the same name. One tending from the centre of curvature, and the other from a fixed centre.

105. Since the normal is always perpendicular to the tangent, the angle between two consecutive normals is equal to $(\delta\phi)$ that between two consecutive tangents;

$$\therefore \delta s = \rho \delta\phi, \quad \text{and} \quad \frac{\delta s}{\delta t} = \rho \cdot \frac{\delta\phi}{\delta t};$$

$$\text{and} \quad \therefore d_t s = \rho d_t \phi,$$

$$\text{wherefore } R = \frac{(\rho d_t \phi)^2}{\rho},$$

$$= \rho (d_t \phi)^2.$$

Hence the centrifugal force spoken of in the last article is equal to the radius of curvature of the *path* at the point where the particle is at a given time, multiplied into the square of the angular velocity of the particle about the centre of curvature: a result which in form coincides with Art. 99, as it ought; for, since the centre of curvature is the intersection of two consecutive normals, the line which joins the particle with

the centre of curvature revolves for an instant about a fixed point. It is this agreement of the two results which renders it less inconvenient than it otherwise would be to apply the same designation to two formulæ which belong to different hypotheses.

106. *The accelerating forces by which a body is urged being all parallel to a fixed line; to obtain the differential equation for the path described.*

Let the axis of y be the line to which the forces are parallel: then since there are no forces parallel to x , we have

$$d_t^2 x = 0, \quad d_t^2 y = Y.$$

The former being integrated gives $d_t x = \text{constant}$. But $d_t x = \text{velocity parallel to the axis of } x$; if then u be the velocity of projection, and α the angle of inclination of its direction to the axis of x ,

$$d_t x = u \cos \alpha.$$

$$\text{Now } d_t y = d_x y \cdot d_t x = u \cos \alpha \cdot d_x y;$$

$$\begin{aligned} \therefore d_t^2 y &= d_t (d_x y) = u \cos \alpha \cdot d_t (d_x y) \\ &= u \cos \alpha \cdot d_x (d_x y) \cdot d_t x \\ &= u^2 \cos^2 \alpha \cdot d_x^2 y; \end{aligned}$$

$$\therefore Y = u^2 \cos^2 \alpha \cdot d_x^2 y.$$

This equation will give the form of the path when the law of force is given: or if the equation of the path be given, we may find the law of force by the action of which it may be described.

Ex. 1. *To find the path described, when the force is a uniform retarding force.*

Here Y is negative and constant; let it $= -f$;

$$\therefore -f = u^2 \cos^2 \alpha \cdot d_x^2 y;$$

$$\therefore -f x = u^2 \cos^2 \alpha \cdot d_x y + \text{constant, by integration.}$$

But supposing the body projected from the origin of co-ordinates, when $x = 0$, $d_x y = \tan \alpha$;

$$\therefore \text{constant} = -u^2 \sin \alpha \cos \alpha;$$

$$\therefore -fx = u^2 \cos^2 \alpha \cdot d_y y - u^2 \sin \alpha \cos \alpha \cdot x;$$

$$\therefore -\frac{1}{2}fx^2 = u^2 \cos^2 \alpha \cdot y - u^2 \sin \alpha \cos \alpha \cdot x + \text{constant}.$$

But when $x = 0$, $y = 0$, and therefore constant = 0;

$$\therefore -\frac{1}{2}fx^2 = u^2 \cos^2 \alpha \cdot y - u^2 \sin \alpha \cos \alpha \cdot x,$$

$$\text{or } y = x \tan \alpha - \frac{f \sec^2 \alpha}{2u^2} \cdot x^2.$$

The reader may compare this equation with Art. 76 (3).

Ex. 2. *To find the force which acting in parallel lines, perpendicular to the principal axis, would cause a body to move in a conic section.*

The general equation of a conic section, taking the principal axis for that of x , is

$$y^2 = 2mx + nx^2;$$

$$\therefore y d_y y = m + nx, \text{ by differentiation.}$$

Differentiating again,

$$y d_y^2 y + (d_y y)^2 = n;$$

$$\therefore y d_y^2 y = n - (d_y y)^2$$

$$= n - \left(\frac{m + nx}{y} \right)^2$$

$$= \frac{ny^2 - (m + nx)^2}{y^3}$$

$$= -\frac{m^2}{y^3};$$

$$\therefore Y = u^2 \cos^2 \alpha \cdot \left(-\frac{m^2}{y^3} \right)$$

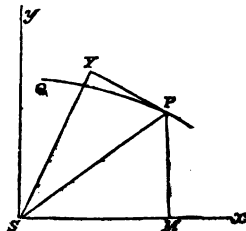
$$= -\frac{m^2 u^2 \cos^2 \alpha}{y^3}.$$

Hence the force varies inversely as the cube of the ordinate, and, being negative, *retards* the motion parallel to the axis of y .

CENTRAL FORCES.

107. **PROB.** *A particle moving obliquely is attracted towards a fixed point; to determine the differential equation of its path referred to polar co-ordinates.*

The motion will take place in one plane passing through the centre of force and the line of direction of original projection; for there is no force acting out of this plane, and the motion being originally in this plane, the body will continue to move in it. To this plane, as that of xy , let the motion be referred: and take the centre of force S as the origin of co-ordinates.



Let P be the place of the body at the time t from some fixed epoch; PQ the path which it takes from that time; and draw PM perpendicular to Sx . Put $x = SM$, $y = PM$, $r = SP$, $\theta = PSx$, $u = \frac{1}{r}$; F = the accelerating force by which the body at P is urged *towards* S (the force being supposed *attractive*).

The force F is equivalent to $F \cos \theta$, $F \sin \theta$ acting respectively parallel to the axes of x , y : consequently by Art. 92 the equations of motion are

$$-d_t^2 x = F \cos \theta = F \frac{x}{r} \dots\dots(1),$$

$$-d_t^2 y = F \sin \theta = F \frac{y}{r} \dots\dots(2),$$

(the negative signs are used because the resolved parts of F act in the direction of $-x$ and $-y$).

Multiply (1) by y and (2) by x , and take the difference of the results; then

$$x d_t^2 y - y d_t^2 x = 0;$$

which being integrated gives

$$x d_t y - y d_t x = a \text{ constant} = h \text{ suppose.}$$

Now since $x = r \cos \theta$, and $y = r \sin \theta$, we have, by differentiation,

$$d_t x = d_t r \cdot \cos \theta - r \sin \theta \cdot d_t \theta = \frac{x}{r} d_t r - y d_t \theta,$$

$$d_t y = d_t r \cdot \sin \theta + r \cos \theta \cdot d_t \theta = \frac{y}{r} d_t r + x d_t \theta;$$

$$\therefore h = r^2 d_t \theta \dots (3) \text{ by substitution.}$$

By means of this we can eliminate t from one of the equations of motion.

$$\begin{aligned} \text{For } d_t x &= d_\theta x \cdot d_t \theta = \frac{h}{r^2} \cdot d_\theta x \\ &= \frac{h}{r^2} \cdot d_\theta (r \cos \theta) \\ &= \frac{h}{r^2} \cdot (d_\theta r \cdot \cos \theta - r \sin \theta) \\ &= -h (d_\theta u \cdot \cos \theta + u \sin \theta); \\ \therefore u &= \frac{1}{r} \text{ and } d_\theta u = -\frac{d_\theta r}{r^2}. \end{aligned}$$

Differentiating again, we have

$$\begin{aligned} d_t^2 x &= -h \cdot d_t (d_\theta u \cdot \cos \theta + u \sin \theta) \\ &= -h \cdot d_\theta (d_\theta u \cdot \cos \theta + u \sin \theta) \cdot d_t \theta \\ &= -h (d_\theta^2 u \cdot \cos \theta + u \cos \theta) \cdot \frac{h}{r^2} \\ &= -h^2 u^2 \cos \theta (d_\theta^2 u + u). \end{aligned}$$

$$\text{But } d_t^2 x = -F \cos \theta;$$

$$\therefore F = h^2 u^2 (d_\theta^2 u + u) \dots (4).$$

When F is known in terms of r or u , this equation will give the relation between r and θ by integration; and then the form of the orbit will be known.

Conversely when the form of the orbit, or the equation between u and θ , is given, this equation will give the law of force by which the orbit may be described.

108. *To find the actual velocity of the body at P.*

Let s denote the length of its path described in the time t ; and v = the velocity at P , then

$$\begin{aligned}
 v^2 &= (d_t s)^2 \\
 &= (d_t x)^2 + (d_t y)^2 \dots \dots \text{Diff. Calc.} \\
 &= (d_t r \cdot \cos \theta - r \sin \theta \cdot d_t \theta)^2 + (d_t r \cdot \sin \theta + r \cos \theta \cdot d_t \theta)^2 \\
 &= (d_t r)^2 + r^2 (d_t \theta)^2 \\
 &= \{(d_\theta r)^2 + r^2\} \cdot (d_t \theta)^2 \\
 &= h^2 \left\{ \left(\frac{d_\theta r}{r^2} \right)^2 + \frac{1}{r^2} \right\} \dots \dots \text{Art. 107 (3)} \\
 &= h^2 \{(d_\theta u)^2 + u^2\}.
 \end{aligned}$$

Now if PY be a tangent to the orbit at P , and SY be perpendicular to it, then

$$\begin{aligned}
 \frac{1}{p^2} &= (d_\theta u)^2 + u^2 \dots \dots \text{Diff. Calc.}; \\
 \therefore v^2 &= \frac{h^2}{p^2}; \\
 \therefore v &= \frac{h}{p}.
 \end{aligned}$$

Hence the velocity at different points of the orbit is inversely proportional to the respective perpendiculars on the tangents at those points.

109. *Whatever be the form of the orbit which the body describes, its velocity and the acute angle, which its course makes with the radius vector, will always be the same at all equal distances from the centre.*

$$\text{For } \therefore v^2 = (d_t x)^2 + (d_t y)^2,$$

$$\therefore v d_t v = d_t x \cdot d_t^2 x + d_t y \cdot d_t^2 y \dots \dots \text{by differentiation}$$

$$= -F \cdot \frac{x d_t x}{r} - F \cdot \frac{y d_t y}{r} \dots \dots \text{Art. 107 (1) (2)}$$

$$= -F d_t r, \quad \therefore r d_t r = x d_t x + y d_t y;$$

$$\therefore v d_r v = -F \dots (1);$$

$$\therefore v^2 = C - 2 \int_r F.$$

Now since F is a function of r , $\therefore \int_r F$ is also a function of r , denote it by $\psi(r)$;

$$\therefore v^2 = C - 2\psi(r) \dots (2).$$

From this equation it appears that the velocity of the body in its orbit depends only on its distance from the centre of force, and not at all on θ , which proves the first part of the proposition.

Again, $p = \frac{h}{v}$; and hence we have equal values of p as well as of v at all those points of the orbit which have the same value of r : consequently at all such points,

$$\frac{p}{r} \text{ is constant.}$$

$$\text{But } \frac{p}{r} = \frac{SY}{SP} = \sin SPY = \sin (180^\circ - SPY).$$

Hence that angle of the two SPY and $180^\circ - SPY$ which is acute, has a constant value, at such points: which proves the latter part of the proposition.

110. DEF. An *apse* is that point in an orbit at which the radius vector is a *normal*.

The analytical character of such a point is that $p = r$,

$$\text{and } \therefore u^2 = \frac{1}{p^3} = u^2 + (d_\theta u)^2;$$

$$\therefore 0 = d_\theta u;$$

$$\therefore \text{also } 0 = d_\theta r.$$

111. DEF. That radius vector which passes through an apse is called an *apsidal distance*, or an *apsidal line*.

It will be shewn presently, that how many apses soever there may be in an orbit, there can only be *two* different apsidal distances at the most. Those apses which are at the greater apsidal distance are called *farther* apses; the others, *nearer* apses.

112. DEF. The angle described by the radius vector about S while the body passes from one apse to the next is called the *apsidal angle*.

113. PROP. *Every apsidal line divides the orbit into two equal, similar, and symmetrical parts.*

For, at a *farther* apse the body which had been *receding* from the centre begins to approach *nearer* to it again; and at a *nearer* apse after having *approached* towards the centre it begins to *recede*. Hence, whether the apse be of the former or the latter kind, two *equal* radii vectores can be found on different sides of it: at those two points the body has equal velocities, equal angular velocities, and equal values of p ; and the acute angle which its course makes with the radius vector is the same. In fact there is but one circumstance which is different at the two points, which is, that the direction of the motion makes with the radius vector an *acute* angle in one case, and an obtuse angle in the other. This circumstance does not however affect the point to be established, which now depends on geometrical properties only. For, corresponding to equal values of r we have equal values of p , and therefore of $d_\theta u$ or of $d_\theta r$, neglecting the sign; and equal values of F , and therefore of $d_\theta^2 u$, or of $d_\theta^2 r$. Hence the curvature is the same at the two points: and the deflections from the tangents are equal. Consequently if the two parts of the orbit which lie on different sides of the apse were to be generated by two points setting out from the apse together, they must needs generate equal, similar, and symmetrical branches, being always equally deflected from the tangent, and always having equal values of p and of r , and always moving with equal angular velocities about S .

Hence the orbit is similar and equal in every respect on each side of an apsidal line.

114. PROP. *There can only be two different apsidal distances at the most.*

For, the body after leaving an apse where r is a maximum must next come, if there be another, to an apse where r is a minimum: and conversely. But since the orbit is similar and equal on each side of the second apse, the body will next

come to an apse exactly similar in every respect to the first; so after leaving this it must next come to an apse similar in every respect to the second and so on. Hence there can only be two different apsidal distances at most.

115. Hence all the apsidal angles are equal. And, if there be but *two* apses, the apsidal angle must be equal to $n \cdot 180^\circ$, n being some integer.

If there be more than *one* apse there must be an *even* number. And if there be four apses, the apsidal angle must be $n \cdot 90^\circ$; n being an integer.

In the preceding articles concerning apses the reasoning supposes them to be such points in the orbit as satisfy the condition of r being a maximum or minimum. Hence those orbits which have asymptotic circles, or pass through the centre of force, will in general be exceptions to what is above said.

116. Hence no orbit can be described round a fixed centre of force, the force being a function of the distance,

1. If the number of normals which can be drawn from that centre of force to the curve, be an *odd* integer greater than 1.
2. If more than *two unequal* normals can be drawn from that point.
3. If each one of such normals do not divide the curve into two equal, similar, and symmetrical parts.
4. If the centre of force be a point in the *evolute*, except in the case of the circle.

117. PROP. *The velocity of the body in its orbit is that due to $\frac{1}{2}$ of the chord of curvature passing through the centre of force.*

For, multiply the equations of motion Art. 107, by $d_t y$, $d_t x$; and take the difference of the results, then

$$d_t x d_t^2 y - d_t y d_t^2 x = F \cdot \frac{x d_t y - y d_t x}{r}.$$

But if ρ be the radius of curvature at the point P ,

$$\rho = \frac{\{(d_t x)^2 + (d_t y)^2\}^{\frac{3}{2}}}{d_t x d_t^2 y - d_t y d_t^2 x} \dots \dots \text{Diff. Calc. and Art. 102,}$$

$$= \frac{v^3}{d_t x d_t^2 y - d_t y d_t^2 x}; \quad \because v^2 = (d_t x)^2 + (d_t y)^2,$$

$$\text{and } x d_t y - y d_t x = h \dots \dots \text{Art. 107,}$$

$$= p v \dots \dots \text{Art. 108;}$$

$$\therefore \frac{v^3}{\rho} = F \cdot \frac{p v}{r};$$

$$\therefore v^2 = F \cdot \frac{\rho p}{r} = 2F \cdot \frac{1}{4} (2\rho \cos PSY)$$

$$= 2F \cdot \frac{1}{4} \text{ chord of curvature.}$$

But if S be the space due (Art. 63) to the velocity v , by the action of the force F continued uniform,

$$v^2 = 2F \cdot S;$$

$$\therefore S = \frac{1}{4} \text{ chord of curvature.}$$

118. To obtain an expression for the force at any point in terms of r and p .

$$\text{Since } \frac{h^2}{p^2} = v^2 = F \frac{\rho p}{r}$$

$$= F \cdot p d_p r; \quad \because \rho = r d_p r;$$

$$\therefore F = \frac{h^2 d_r p}{p^3}$$

$$= -\frac{h^2}{2} \cdot d_r \left(\frac{1}{p^2} \right).$$

119. As an example of the application of this formula, let it be proposed to find the law of force, situated in the centre of an ellipse, by the action of which a body may have the ellipse for its orbit.

The equation of an ellipse between r and p , when the centre is the origin of co-ordinates, is

$$\frac{1}{p^2} = \frac{a^2 + b^2 - r^2}{a^2 b^2}, \dots\dots \text{Conic Sections,}$$

a, b being the semi-axes;

$$\therefore d_r \left(\frac{1}{p^2} \right) = - \frac{2r}{a^2 b^2};$$

$$\therefore F = \frac{h^2}{a^2 b^2} \cdot r, \text{ which varies as } r.$$

120. As an example of the application of the formula (4) in Art. 107, take the following;

To find the law of force, situated in the focus, by which a body may describe an ellipse.

In this case the equation of the ellipse is,

$$u = \frac{1}{r} = \frac{1 + e \cos \theta}{a(1 - e^2)}; \dots\dots \text{Conic Sections;}$$

$$\therefore d_\theta^2 u = - \frac{e \cos \theta}{a(1 - e^2)};$$

$$\therefore d_\theta^2 u + u = \frac{1}{a(1 - e^2)} = \frac{a}{b^2};$$

$$\therefore F = h^2 u^2 \cdot \frac{a}{b^2} \dots\dots \text{Art. 107 (4)}$$

$$= \frac{a h^2}{b^2 r^2},$$

which varies as $\frac{1}{r^2}$.

121. PROP. *If a body be attracted towards several centres of force, all situated in the plane of motion, and*

$v_1, v_2, v_3, \dots$ be the velocities due to $\frac{1}{4}$ of each of the chords of curvature passing through the respective centres, then the (vel.)² = $v_1^2 + v_2^2 + v_3^2 + \dots$

For, take any fixed point in the plane of motion for the origin of co-ordinates. Let $F_1, F_2, F_3, \dots$ be the forces; $a_1b_1, a_2b_2, a_3b_3, \dots$ the co-ordinates of their centres; $\chi_1, \chi_2, \chi_3, \dots$ the chords of curvature passing through them. Then

$$v_1^2 = 2F_1 \cdot \frac{\chi_1}{4}, \quad v_2^2 = 2F_2 \cdot \frac{\chi_2}{4}, \quad v_3^2 = 2F_3 \cdot \frac{\chi_3}{4}, \dots$$

Also if $r_1, r_2, r_3, \dots$ be the distances of the body from the centres of force, the equations of motion are

$$d_t^2 x = -F_1 \frac{x-a_1}{r_1} - F_2 \frac{x-a_2}{r_2} - F_3 \frac{x-a_3}{r_3} - \dots (1),$$

$$d_t^2 y = -F_1 \frac{y-b_1}{r_1} - F_2 \frac{y-b_2}{r_2} - F_3 \frac{y-b_3}{r_3} - \dots (2).$$

Now let $p_1, p_2, p_3, \dots$ be the perpendiculars from the centres of force upon the tangent to the orbit; and let s be the length of the arc described in the time t , and ρ the radius of curvature.

Multiply (1) by $d_t y$ and (2) by $d_t x$ and subtract the latter result from the former.

The numerator of the fraction which has r_1 for its denominator is

$$\begin{aligned} (x-a_1) d_t y - (y-b_1) d_t x &= (x-a_1) d_t (y-b_1) - (y-b_1) d_t (x-a_1) \\ &= p_1 d_t s, \text{ by Differential Calculus.} \end{aligned}$$

Hence

$$d_t x d_t^2 y - d_t y d_t^2 x = F_1 \frac{p_1 d_t s}{r_1} + F_2 \frac{p_2 d_t s}{r_2} + F_3 \frac{p_3 d_t s}{r_3} + \dots;$$

$$\therefore \frac{v^3}{\rho} = (F_1 \frac{p_1}{r_1} + F_2 \frac{p_2}{r_2} + F_3 \frac{p_3}{r_3} + \dots) v;$$

$$\begin{aligned}
 \therefore v^2 &= F_1 \frac{\rho p_1}{r_1} + F_2 \frac{\rho p_2}{r_2} + F_3 \frac{\rho p_3}{r_3} + \dots \\
 &= 2 F_1 \cdot \frac{\chi_1}{4} + 2 F_2 \cdot \frac{\chi_2}{4} + 2 F_3 \cdot \frac{\chi_3}{4} + \dots \\
 &= v_1^2 + v_2^2 + v_3^2 + \dots
 \end{aligned}$$

122. *If a body moving in an orbit describe about some point areas proportional to the times of their description, that point is the centre of force.*

For, make it the origin of co-ordinates, and let $A (= Ct)$ be the area described in the time t ; then

$$d_\theta A = \frac{1}{2} r^2 \dots \text{by Differential Calculus};$$

$$\therefore 2 d_t (Ct) = r^2 d_t \theta,$$

$$\text{or } 2C = x d_t y - y d_t x.$$

Differentiating again,

$$0 = x d_t^2 y - y d_t^2 x;$$

$$\therefore \frac{d_t^2 x}{d_t^2 y} = \frac{x}{y} = \frac{SM}{MP} \text{ (fig. of Art. 107).}$$

That is, the forces in direction of the co-ordinate axes are proportional to the co-ordinates; we may therefore take these co-ordinates SM , MP to represent the forces: these are equivalent to a single force acting in the line PS , which always passes through S ; therefore S is the centre of force.

123. From the equations and properties which have been investigated in the preceding Articles on the motion of a body subject to the action of a central force, we may find either the law of force by the action of which a given orbit might be described; or the nature of the orbit which would be described by the action of a given force. We have given examples of the former process, we shall now subjoin examples of the latter kind. Before doing this, however, it may be useful to collect into one view from the preceding articles the equations which are necessary for our purpose.

If merely the nature of the orbit be required, it may be found by one integration of the equation in Art. 118,

$$F = -\frac{h^2}{2} \cdot d_r \left(\frac{1}{p^2} \right),$$

which will give the relation between r and p .

But if, under given initial circumstances, we want to know not merely the nature but the exact magnitude and position of the orbit described, we may employ the three equations

$$h = pv, \quad v d_r v = -F,$$

$$u^2 + (d_\theta u)^2 \left(= \frac{1}{p^2} \right) = \frac{v^2}{h^2};$$

or instead of one of these we may use the equation

$$F = h^2 u^2 (u + d_\theta^2 u).$$

A better way than this, whenever it can be employed, is, after having found the *nature* of the orbit by the first case, to assume, if it be known, the equation of the orbit between r and θ , and then determine from the initial circumstances the values of the constants in it. Examples of both methods will be given.

If the place of the body at any time be required it must be found by integrating the equation

$$d_\theta t = \frac{r^2}{h},$$

the constant introduced by integration being determined from the given circumstances.

124. *The force varying as the distance, to determine the nature of the orbit.*

Here because $F \propto r$, let it $= \mu r$; μ being the absolute force;

$$\therefore \frac{h^2}{2} \cdot d_r \left(\frac{1}{p^2} \right) = -F = -\mu r;$$

$$\therefore \frac{h^2}{p^2} = C - \mu r^2 \dots \dots \dots (1).$$

Now the equation of an ellipse about the centre as pole is,

$$\frac{1}{p^2} = \frac{a^2 + b^2 - r^2}{a^2 b^2};$$

$$\therefore \frac{h^2}{p^3} = \left(\frac{h^2}{a^2} + \frac{h^2}{b^2} \right) - \frac{h^2}{a^2 b^2} \cdot r^2.$$

This coincides exactly in form with the equation (1) of the orbit; and hence the orbit is an ellipse, whose centre coincides with the centre of force..

125. *To find the periodic time*; that is, the time between the body leaving any point in the orbit and returning thither again.

By comparing the like parts of the last equation with (1), we have

$$\mu = \frac{h^2}{a^2 b^2}, \quad \text{or } h^2 = a^2 b^2 \mu.$$

Now the area described in one second is equal to $\frac{h}{2}$, and the area being proportional to the time, we have

$$\begin{aligned} \text{Periodic time} &= \frac{\text{area of orbit}}{\text{area described in 1''}} \\ &= \frac{2 \text{ area of the ellipse}}{h} \\ &= \frac{2\pi ab}{ab\sqrt{\mu}} \\ &= \frac{2\pi}{\sqrt{\mu}}. \end{aligned}$$

As this result is obtained without taking into account either the velocity, direction, or distance of projection, it must be true whatever these should happen to be.

And because it involves neither a nor b , it is independent of the size of the orbit.

Hence, when the force varies as the distance, whatever be the velocity, direction, and distance of projection, or what-

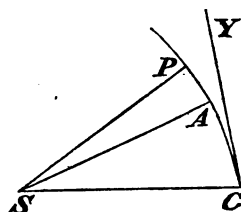
ever be the magnitude and position of the orbit, the body will always return to the point from which it started in the same time.

126. **PROB.** *The body being projected from a given point, with a given velocity and in a given direction, and being acted on by a force varying as the distance; it is required to determine the magnitude and position of the ellipse which it will describe.*

Let C be the point from which the body is projected, CY the direction of projection; P the place of the body at any time t after projection. Put $SC = R$, velocity of projection $= V$, and the $\angle SCY = \alpha$,

$$\text{Then } \frac{h^2}{2} d_r \left(\frac{1}{p^2} \right) = -\mu r,$$

$$\text{and } \therefore v^2 = \frac{h^2}{p^2} = C - \mu r^2.$$



But at the point of projection

$$v = V, \quad r = R, \quad \text{and } \therefore C = V^2 + \mu R^2;$$

$$\text{hence } \frac{h^2}{p^2} = V^2 + \mu R^2 - \mu r^2 \dots\dots (1).$$

Now the equation of an ellipse about the centre as pole gives

$$\frac{a^2 b^2}{p^2} = a^2 + b^2 - r^2,$$

and multiplying this by μ , we have

$$\frac{\mu a^2 b^2}{p^2} = \mu (a^2 + b^2) - \mu r^2;$$

which being compared with equation (1) gives

$$a^2 + b^2 = \frac{V^2}{\mu} + R^2 \dots\dots (2).$$

The form of this equation shews that $\frac{V}{\sqrt{\mu}}$ is the semi-diameter conjugate to R ; and consequently by the property of parallelograms, whose sides are tangents to an ellipse at the extremities of conjugate diameters, being all of the same area,

$$ab = \frac{V}{\sqrt{\mu}} \cdot R \cdot \sin \alpha \dots (3);$$

(2) and (3) determine the *magnitude* of the orbit. Let AS be the position of the major axis, and put $\varpi = CSA$. Then because S is the centre of the ellipse,

$$\frac{(R \cos \varpi)^2}{a^2} + \frac{(R \sin \varpi)^2}{b^2} = 1;$$

from which eliminating a and b by means of (2) and (3), we find

$$\cot 2\varpi = \cot 2\alpha + \frac{\mu R^2}{V^2} \operatorname{cosec} 2\alpha,$$

which giving the value of ϖ determines the *position* of the orbit described.

127. To determine the place of the body in its orbit at a given time.

Let $\theta = CSP$, the angle described by the body in the time t : then the equation of the ellipse gives

$$\frac{r^2 \cos^2 (\theta - \varpi)}{a^2} + \frac{r^2 \sin^2 (\theta - \varpi)}{b^2} = 1.$$

$$\begin{aligned} \text{Now } d_\theta t &= \frac{r^2}{h} = \frac{1}{h} \cdot \frac{a^2 b^2}{b^2 \cos^2 (\theta - \varpi) + a^2 \sin^2 (\theta - \varpi)} \\ &= \frac{ab}{\sqrt{\mu}} \cdot \frac{\sec^2 (\theta - \varpi)}{b^2 + a^2 \tan^2 (\theta - \varpi)} \\ &= \frac{b}{\sqrt{\mu}} \cdot \frac{a d_\theta \tan (\theta - \varpi)}{b^2 + a^2 \tan^2 (\theta - \varpi)}; \end{aligned}$$

$$\therefore t = \frac{1}{\sqrt{\mu}} \cdot \tan^{-1} \left(\frac{a}{b} \tan (\theta - \varpi) \right) + \text{const.}$$

$$\text{Now when } t = 0, \theta = 0, \therefore \text{const} = \frac{1}{\sqrt{\mu}} \tan^{-1} \left(\frac{a}{b} \tan \varpi \right);$$

$$\therefore t \sqrt{\mu} = \tan^{-1} \left(\frac{a}{b} \tan (\theta - \varpi) \right) + \tan^{-1} \left(\frac{a}{b} \tan \varpi \right).$$

This equation furnishes the value of θ , for any given value of t .

128. **PROB.** *A body is projected from a given point, with a given velocity, in a given direction and is acted on by a central force varying inversely as the square of the distance; to determine the nature, magnitude and position of the orbit which it will describe.*

Using the same figure and notation as in Art. 126, we have $\left(\text{since } F = \frac{\mu}{r^2} \right)$,

$$\frac{h^2}{2} d_r \left(\frac{1}{p^2} \right) = - \frac{\mu}{r^2};$$

$$\therefore v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + \text{constant.}$$

But at the point of projection $v = V, r = R$,

$$\text{and } \therefore \text{const} = V^2 - \frac{2\mu}{R};$$

$$\text{hence } \frac{h^2}{p^2} = \frac{2\mu}{r} + V^2 - \frac{2\mu}{R} \dots\dots(1).$$

Now the equation of a conic section about a focus as origin is

$$\frac{b^2}{p^2} = \frac{2a}{r} \mp 1, \text{ or } \frac{2a}{r};$$

the last term being $-1, 0, +1$, according as this conic section

is an ellipse, a parabola, or a hyperbola : multiplying this by $\frac{\mu}{a}$, we have

$$\frac{\mu b^2}{ap^2} = \frac{2\mu}{r} \mp \frac{\mu}{a}, \text{ or } \frac{2\mu}{r}.$$

This being compared with equation (1) shews that

if $V^2 < \frac{2\mu}{R}$ the orbit is an *ellipse*,

if $V^2 = \frac{2\mu}{R}$ a *parabola*,

if $V^2 > \frac{2\mu}{R}$ a *hyperbola* :

and taking the ellipse as a standard case, to avoid ambiguities of sign,

$$\frac{\mu}{a} = \frac{2\mu}{R} - V^2 \dots\dots\dots (2),$$

$$\text{and } \frac{\mu b^2}{a} = h^2 = V^2 R^2 \sin^2 \alpha \dots (3),$$

(for $h = vp = VR \sin \alpha$ at the point of projection).

(2) and (3) give successively the values of a and b , which determine the *magnitude* of the orbit. The *position* of it may be thus found.

Let e denote the eccentricity of the orbit, which is known from (2) and (3); then from the polar equation of an ellipse, at the point of projection

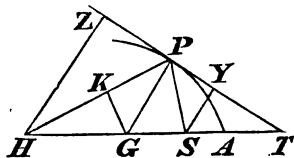
$$\begin{aligned} R &= \frac{b^2}{a(1 + e \cos \varpi)} \\ &= \frac{V^2 R^2 \sin^2 \alpha}{\mu(1 + e \cos \varpi)} \text{ from (3);} \end{aligned}$$

from which ϖ is known.

129. If we attempt to find the position of the apse from the last equation ambiguities of sign are involved in the pro-

cess, which may be avoided by another method of solution. The whole problem also admits of a simple solution on the principles derived from Art. 102: which, therefore, and also because it does not involve any process of integration, we shall lay before the reader.

Let S be the centre of force, and suppose the body projects from the point P , at the distance $SP = R$, with a velocity V , in the direction PZ making an angle $SPZ = \alpha$ with SP . Let H be the other focus of the ellipse: join HS and produce it and ZP to meet in T ; draw PG , HZ , SY perpendicular to TZ ; and KG to HP . Then the vertex A of the orbit AP is in ST . Let a , b , e be as before, and $\varpi =$ the angle PSA .



By Art. 102, the normal force at $P = \frac{(\text{velocity})^2}{\text{rad. curv.}} \dots\dots(1).$

But normal force at $P = \frac{\mu}{R^2} \cos SPG = \frac{\mu}{R^2} \sin \alpha$;

and $\text{rad. curv. at } P = \frac{(\text{normal})^3}{(\frac{1}{2} \text{ lat. rect.})^2}.$

But $\text{normal } PG = \frac{PK}{\cos GPH} = \frac{\frac{1}{2} \text{ lat. rect.}}{\sin \alpha}$;

$\therefore \text{rad. curv.} = \frac{\frac{1}{2} \text{ lat. rect.}}{\sin^3 \alpha} = \frac{b^2}{a \sin^3 \alpha}.$

Hence, making substitution in equation (1), we find

$$\frac{\mu \sin \alpha}{R^2} = \frac{V^2 a \sin^3 \alpha}{b^2}$$

$$\text{or } \frac{b^2}{a} = \frac{R^2 V^2 \sin^2 \alpha}{\mu} \dots\dots(2).$$

Again, since $b^2 = SY \cdot HZ = SP \sin \alpha \cdot HP \sin \alpha$
 $= R (2a - R) \sin^2 \alpha,$

this being substituted in equation (2) gives

$$\frac{\mu}{a} = \frac{2\mu}{R} - V^2 \dots \dots \dots (3).$$

Also $e^2 = 1 - \frac{b^2}{a^2}$

$$= 1 - \frac{R^2 V^2 \sin^2 \alpha}{\mu^2} \left(\frac{2\mu}{R} - V^2 \right)$$

$$= \cos^2 \alpha + \left(\frac{V^2 R}{\mu} - 1 \right)^2 \sin^2 \alpha \dots \dots (4).$$

And $\tan (\alpha - \varpi) = \tan PTS$

$$= \frac{SH \sin T}{SH \cos T}$$

$$= \frac{HZ - SY}{PZ + PY}$$

$$= \frac{(HP - SP) \sin (180^\circ - \alpha)}{(HP + SP) \cos (180^\circ - \alpha)}$$

$$= - \frac{2a - 2R}{2a} \tan \alpha$$

$$= \left(\frac{R}{a} - 1 \right) \tan \alpha$$

$$= \left(1 - \frac{RV^2}{\mu} \right) \tan \alpha, \text{ from (3);}$$

$$\tan \varpi = \frac{-\sin \alpha \cos \alpha}{\sin^2 \alpha - \frac{\mu}{RV^2}} \dots \dots \dots (5).$$

The equations (2) (3) (4) (5) determine the magnitude and position of the orbit.

130. By substituting for a its value from (3), we find

$$HP = 2a - R = \frac{R}{\frac{2\mu}{RV^2} - 1} \dots\dots(6).$$

Now the line PH is given in position (for SP , HP make equal angles with YZ); this equation will therefore fix the position of the major axis without the assistance of (5).

131. The equations found in the two preceding articles will enable us to see what is the nature of the effect which any small extraneous force tends to produce upon the orbit. (An extraneous force is one which does not emanate from S , or if it be at S , does not $\propto \frac{1}{D^2}$).

Thus, if the motion be performed in a resisting medium of small density, since the effect of such a medium is to diminish the velocity, we learn from equation (3) that in consequence of such a resistance the major axis would decrease perpetually; and from equation (6) that HP would be diminished, and therefore H shifted nearer to P , which shews that the major axis would *regrede* as P moves towards farther apse, and *progrede* during the other half of P 's revolution. We may learn the effect upon the eccentricity from (4). For, (3) may be written $\frac{R}{a} - 1 = \frac{V^2 R}{\mu} - 1$; and therefore the value of $\left(\frac{V^2 R}{\mu} - 1\right)^2$, being equal to $\left(\frac{R}{a} - 1\right)^2$, increases or decreases simultaneously with $(R - a)^2$; that is, as P passes from an extremity of the minor axis to an apse, the eccentricity is *increased* by resistance; and as P passes from an apse to an extremity of the minor axis, the eccentricity is *diminished* by resistance.

132. Equation (3) not containing a shews that the length of the major axis is independent of the *direction* in which the body is projected.

The equation for b^2 which precedes (3) shews that *cæteris paribus* the length of the minor axis is proportional to the sine of the angle of projection.

We may find the conditions of projection that the orbit may be a circle by making $e = 0$ in equation (4); the right-hand member of the equation being the sum of two squares, the result is equivalent to the two conditions

$$\cos \alpha = 0, \text{ and } \frac{V^2 R}{\mu} - 1 = 0;$$

$$\text{or } \alpha = 90^\circ, \text{ and } V^2 = \frac{\mu}{R}.$$

133. When the orbit is neither parabolic nor hyperbolic, the periodic time may be readily found as in Art. 125.

$$\begin{aligned} \text{For, period} &= \frac{2 \text{ area of ellipse}}{h} \\ &= \frac{2 \pi a b}{\left(\frac{b^2 \mu}{a}\right)^{\frac{1}{2}}} \text{ from Art. 128, (3)} \\ &= \frac{2 \pi a^{\frac{3}{2}}}{\sqrt{\mu}}, \end{aligned}$$

which depending only on the major axis is independent of the direction of projection.

134. **PROB.** *When a body moves in a parabolic orbit about a force in the focus varying as $\frac{1}{D^2}$, to find the time of describing any given angle from leaving the apse.*

Since the orbit is a parabola $V^2 = \frac{2\mu}{R}$ (Art. 128): at the apse, therefore, we have $h^2 = V^2 R^3 \sin^2 \alpha = 2\mu R = 2\mu m$, $4m$ denoting the latus rectum of the orbit.

$$\text{Also the equation of the orbit is } r = \frac{m}{\cos^2 \frac{\theta}{2}} = m \sec^2 \frac{\theta}{2};$$

$$\begin{aligned}
 \therefore d_\theta t &= \frac{r^2}{h} = \frac{m^2}{\sqrt{2\mu m}} \cdot \sec^4 \frac{\theta}{2} \\
 &= \frac{m^{\frac{1}{2}}}{\sqrt{2\mu}} \cdot \sec^2 \frac{\theta}{2} \cdot 2d_\theta \tan \frac{\theta}{2} \\
 &= m^{\frac{1}{2}} \left(\frac{2}{\mu}\right)^{\frac{1}{2}} \cdot \left(1 + \tan^2 \frac{\theta}{2}\right) d_\theta \tan \frac{\theta}{2}; \\
 \therefore t &= m^{\frac{1}{2}} \left(\frac{2}{\mu}\right)^{\frac{1}{2}} \left(\tan \frac{\theta}{2} + \frac{1}{3} \tan^3 \frac{\theta}{2}\right).
 \end{aligned}$$

No constant is added, because by the question $t = 0$ when $\theta = 0$.

135. PROB. *When a body describes an elliptical orbit about a force in one of its foci varying as $\frac{1}{D^3}$, to find the time of describing any given angle from nearer apse.*

$$\text{Here } r = \frac{b^2}{a(1 + e \cos \theta)};$$

$$\therefore d_\theta t = \frac{r^2}{h} = \frac{b^4}{a^2 h} \cdot \frac{1}{(1 + e \cos \theta)^2};$$

$$\therefore t = \frac{b^4}{a^2 h} \cdot \int_\theta \frac{1}{(1 + e \cos \theta)^2}.$$

$$\text{Assume } \int_\theta \frac{1}{(1 + e \cos \theta)^2} = \frac{A \sin \theta}{1 + e \cos \theta} + \int_\theta \frac{B}{1 + e \cos \theta},$$

then differentiating both sides, and reducing the results to a common denominator and equating the numerators, we have

$$1 = (A + Be) \cos \theta + Ae + B;$$

$$\therefore 0 = A + Be, \text{ and } 1 = Ae + B;$$

$$\text{whence } A = \frac{-e}{1 - e^2} \text{ and } B = \frac{1}{1 - e^2};$$

$$\therefore t = \frac{b^4}{a^2 h} \left\{ \frac{-e \sin \theta}{(1 - e^2)(1 + e \cos \theta)} + \frac{1}{1 - e^2} \cdot \int_\theta \frac{1}{1 + e \cos \theta} \right\}.$$

$$\begin{aligned}
 \text{Again, } \int_0^\theta \frac{1}{1 + e \cos \theta} &= \int_0^\theta \frac{1}{\left(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}\right) + e \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}\right)} \\
 &= \int_0^\theta \frac{\sec^2 \frac{\theta}{2}}{(1 + e) + (1 - e) \tan^2 \frac{\theta}{2}} \\
 &= 2 \cdot \int_0^\theta \frac{d_\theta \tan \frac{\theta}{2}}{(1 + e) + (1 - e) \tan^2 \frac{\theta}{2}} \\
 &= \frac{2}{\sqrt{1 - e^2}} \cdot \tan^{-1} \left(\sqrt{\frac{1 - e}{1 + e}} \cdot \tan \frac{\theta}{2} \right); \\
 \therefore t &= \frac{b^2}{h} \cdot \frac{-e \sin \theta}{1 + e \cos \theta} + \frac{2ab}{h} \cdot \tan^{-1} \left(\sqrt{\frac{1 - e}{1 + e}} \cdot \tan \frac{\theta}{2} \right).
 \end{aligned}$$

No arbitrary constant is added to this integral, because when $t = 0$, $\theta = 0$, by the question.

From Art. 128, (3) we have

$$h = b \left(\frac{\mu}{a} \right)^{\frac{1}{2}};$$

$$\therefore t = \frac{b\sqrt{a}}{\sqrt{\mu}} \cdot \frac{-e \sin \theta}{1 + e \cos \theta} + \frac{2a^{\frac{3}{2}}}{\sqrt{\mu}} \cdot \tan^{-1} \left(\sqrt{\frac{1 - e}{1 + e}} \cdot \tan \frac{\theta}{2} \right).$$

136. If the orbit had been an hyperbola, the process would have been the same as far as the last integration, when we should have, in consequence of e being > 1 ,

$$\begin{aligned}
 \int_0^\theta \frac{1}{1 + e \cos \theta} &= 2 \int_0^\theta \frac{d_\theta \tan \frac{\theta}{2}}{(e + 1) - (e - 1) \tan^2 \frac{\theta}{2}} \\
 &= \frac{1}{\sqrt{e^2 - 1}} \cdot \log_e \frac{\sqrt{e + 1} + \sqrt{e - 1} \cdot \tan \frac{\theta}{2}}{\sqrt{e + 1} - \sqrt{e - 1} \cdot \tan \frac{\theta}{2}};
 \end{aligned}$$

$$\therefore t = \frac{b\sqrt{a}}{\sqrt{\mu}} \cdot \frac{e \sin \theta}{1 + e \cos \theta} - \frac{a^{\frac{1}{2}}}{\sqrt{\mu}} \cdot \log \frac{\sqrt{e+1} + \sqrt{e-1} \cdot \tan \frac{\theta}{2}}{\sqrt{e+1} - \sqrt{e-1} \cdot \tan \frac{\theta}{2}}.$$

137. PROB. *To find the orbit when the force varies inversely as the 5th power of the distance.*

$$\text{Here } F = \frac{\mu}{r^5};$$

$$\therefore v dv = -F = -\frac{\mu}{r^5};$$

$$\therefore v^2 = \frac{\mu}{2r^4} + C.$$

Now at the point of projection $v = V$, $r = R$;

$$\therefore C = V^2 - \frac{\mu}{2R^4},$$

$$v^2 = \frac{\mu}{2r^4} + V^2 - \frac{\mu}{2R^4};$$

$$\therefore \frac{h^2}{p^2} = \frac{\mu}{2r^4} + V^2 - \frac{\mu}{2R^4}.$$

But $h = VR \sin \alpha$;

$$\therefore \frac{V^2 R^2 \sin^2 \alpha}{p^2} = \frac{\mu}{2r^4} + V^2 - \frac{\mu}{2R^4};$$

$$\therefore V^2 R^2 \sin^2 \alpha \{u^2 + (d_\theta u)^2\} = \frac{\mu}{2} \cdot u^4 + V^2 - \frac{\mu}{2R^4};$$

$$\therefore V^2 R^2 \sin^2 \alpha \cdot (d_\theta u)^2 = \frac{\mu}{2} \cdot u^4 - V^2 R^2 \sin^2 \alpha \cdot u^2 + V^2 - \frac{\mu}{2R^4}.$$

This equation cannot be integrated in finite terms, unless either the right hand member be a complete square, or

$$V^2 - \frac{\mu}{2R^4} = 0.$$

1st. When the right hand member is an exact square.
That this may be the case, we must have

$$2\mu \left(V^2 - \frac{\mu}{2R^4} \right) = V^4 R^4 \sin^4 \alpha,$$

from which the angle α is known.

Hence

$$V^2 R^2 \sin^2 \alpha \cdot (d_\theta u)^2 = \frac{\mu}{2} \cdot u^4 - V^2 R^2 \sin^2 \alpha \cdot u^2 + \frac{V^4 R^4 \sin^4 \alpha}{2\mu};$$

$$\therefore 1 = \frac{VR \sin \alpha \cdot d_\theta u}{\left(\frac{\mu}{2}\right)^{\frac{1}{2}} \cdot u^2 - \frac{V^2 R^2 \sin^2 \alpha}{\sqrt{2\mu}}}$$

$$= -\frac{\sqrt{2\mu} \cdot VR \sin \alpha \cdot d_\theta r}{\mu - V^2 R^2 \sin^2 \alpha \cdot r^2};$$

$$\therefore \theta + \text{const.} = \frac{1}{\sqrt{2}} \cdot \log_e \frac{VR \sin \alpha \cdot r - \sqrt{\mu}}{VR \sin \alpha \cdot r + \sqrt{\mu}}.$$

Let the axis of x be taken in such a position that the const. = 0, then

$$e^{\theta\sqrt{2}} = \frac{VR \sin \alpha \cdot r - \sqrt{\mu}}{VR \sin \alpha \cdot r + \sqrt{\mu}};$$

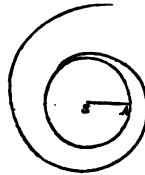
$$\therefore r = \frac{\sqrt{\mu}}{VR \sin \alpha} \cdot \frac{1 + e^{\theta\sqrt{2}}}{1 - e^{\theta\sqrt{2}}}.$$

In extracting the root before integration, if we had prefixed the sign $\pm$, the lower sign would have given

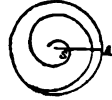
$$r = \frac{\sqrt{\mu}}{VR \sin \alpha} \cdot \frac{e^{\theta\sqrt{2}} - 1}{e^{\theta\sqrt{2}} + 1}.$$

The former is represented by the following figure, which is a spiral line approaching continually nearer to

a circle whose radius = $\frac{\sqrt{\mu}}{VR \sin \alpha}$, which it cannot reach till an infinite number of revolutions have been performed. Such a circle is called an interior *asymptotic circle*.



The latter equation represents the spiral in the following figure, which has an exterior asymptotic circle of the same radius as the former.



2ndly. When $V^2 - \frac{\mu}{2R^4} = 0$.

This falls under the following more general example.

138. **PROB.** *To find the orbit when the force varies inversely as the n^{th} power of the distance, and the velocity of projection is that which a body would acquire in falling directly towards the centre to the point of projection from an infinite distance.*

Let the force $= \frac{\mu}{r^n}$.

To find the velocity acquired in falling from an infinite distance, we must employ the formula $v' d_s v' = -F$ in Art. 87, s being its distance from the centre when its vel. $= v'$;

$$\therefore v' d_s v' = -\frac{\mu}{s^n};$$

$$\therefore v'^2 = \frac{2\mu}{(n-1)s^{n-1}} + C.$$

Now when $s = \infty$, $v' = 0$; $\therefore C = 0$, when n is > 1 , but $C = \infty$ when n is not > 1 ; we shall therefore suppose $n > 1$.

Also, when $s = R$, $v' = V$; $\therefore V^2 = \frac{2\mu}{(n-1)R^{n-1}}$.

To find the orbit we have

$$v d_s v = -F = -\frac{\mu}{r^n};$$

$$\therefore v^2 = \frac{2\mu}{(n-1)r^{n-1}} + C'.$$

At the point of projection $r = R$, and $v = V$; $\therefore C' = 0$;

$$\therefore \frac{h^2}{p^2} = \frac{2\mu}{(n-1)r^{n-1}}.$$

$$\text{But } h^2 = V^2 R^2 \sin^2 \alpha = \frac{2\mu \sin^2 \alpha}{(n-1) \cdot R^{n-3}};$$

$$\therefore \frac{1}{p^2} = \frac{R^{n-3}}{r^{n-1} \sin^2 \alpha}.$$

Now substituting $(d_\theta u)^2 + u^2$ or rather $\frac{(d_\theta r)^2}{r^4} + \frac{1}{r^2}$ for $\frac{1}{p^2}$; we find after reduction,

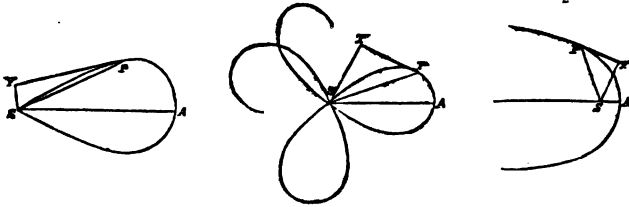
$$1 = \frac{-r^{\frac{n-5}{2}} d_\theta r}{\sqrt{R^{n-3} \operatorname{cosec}^2 \alpha - r^{n-3}}};$$

$$\therefore \theta = \frac{2}{n-3} \cos^{-1} \left(\frac{r^{\frac{n-3}{2}}}{R^{\frac{n-3}{2}} \operatorname{cosec} \alpha} \right) + C''.$$

For simplicity, as we only want to know the *nature* of the orbit, we may suppose the axis of α taken in such a position that $C'' = 0$;

$$\therefore \left(\frac{r}{R} \right)^{\frac{n-3}{2}} = \operatorname{cosec} \alpha \cos \left(\frac{n-3}{2} \theta \right).$$

When n is > 3 , this equation is that of the curve in the first of the following figures, if $\frac{n-3}{2}$ be an even number; but it is that of the second figure, if $\frac{n-3}{2}$ be an odd number.



When n is < 3 but > 1 , the orbit is that in the third figure.

The characteristic property of the orbits of the present article is that the $\angle PSY = \frac{n-3}{2} \cdot \angle ASP$, in the above figures, where SA is the apsidal line.

$$\text{For, } \cos\left(\frac{n-3}{2}\theta\right) \cdot \text{cosec } a = \left(\frac{r}{R}\right)^{\frac{n-3}{2}} = \frac{p}{r} \text{ cosec } a = \cos PSY \cdot \text{cosec } a;$$

$$\therefore \cos\left(\frac{n-3}{2}\theta\right) = \cos PSY;$$

$$\therefore \frac{n-3}{2} \cdot PSA = PSY.$$

When $n = 2$, then $\frac{1}{2} PSA = PSY$, or $PSY = YSA$, (third figure in preceding page,) which is the property of the parabola.

The body will either fly off *in infinitum* or fall into the centre, when it has described an angle from the apse equal to $\frac{\pi}{n-3}$, according as n is less or greater than 3 respectively.

139. The Earth attracts the Moon with a force which varies inversely as the square of her distance from its centre. If this were the only force to which she is subject, her orbit would be an ellipse having one focus in the Earth's centre. It is found by calculation that the Sun exerts a disturbing force, small however in comparison with the force of the Earth, such that to first approximations the Moon upon the whole is urged towards the Earth with a force part of which varies inversely as the square, and part inversely as the cube of her distance. It is required to find the nature of her orbit, neglecting all other disturbing forces.

Let $\frac{\mu}{r^2} (= \mu u^2)$ and $\frac{\mu'}{r^3} (= \mu' u^3)$ be the two parts of the force upon the Moon, μ' being small.

Then by Art. 107

$$\mu u^2 + \mu' u^3 = h^2 u^2 (u + d_0^2 u),$$

$$\text{or } d_0^2 u + \left(1 - \frac{\mu'}{h^2}\right) u = \frac{\mu}{h^2}.$$

For brevity write n^2 for $1 - \frac{\mu'}{h^2}$, and $n^2 a'$ for $\frac{\mu}{h^2}$;

$$\therefore d_\theta^2 u + n^2 u = n^2 a',^*$$

which may be written

$$d_\theta^2 (u - a') + n^2 (u - a') = 0;$$

Multiply by $2d_\theta(u - a')$ and integrate;

$$\begin{aligned} \therefore \{d_\theta(u - a')\}^2 + n^2(u - a')^2 &= \text{constant} \\ &= n^2 C^2 \text{ suppose;} \end{aligned}$$

$$\therefore n = \frac{\pm d_\theta(u - a')}{\sqrt{C^2 - (u - a')^2}}.$$

Taking the lower sign, which is equivalent to supposing θ and r to increase together, or the Moon to be receding from the Earth; a supposition which will not affect the generality of our results; we have by integration

$$n\theta + C' = \cos^{-1} \left(\frac{u - a'}{C} \right).$$

* As equations of this form are of frequent occurrence in physical problems, the reader will find the following general method of integrating them useful.

Take the general form of $d_\theta^2 u + n^2 u = \Theta$, Θ being a given function of θ .

Multiplying by $\cos(n\theta + D)$ and integrating by parts, D being any constant quantity; we have,

$$\int_\theta \{d_\theta^2 u \cos(n\theta + D)\} = d_\theta u \cdot \cos(n\theta + D) + \int_\theta \{n d_\theta u \sin(n\theta + D)\}.$$

Also

$$\int_\theta \{n^2 u \cos(n\theta + D)\} = n u \sin(n\theta + D) - \int_\theta \{n d_\theta u \sin(n\theta + D)\};$$

therefore by addition,

$$\int_\theta \Theta \cos(n\theta + D) = d_\theta u \cdot \cos(n\theta + D) + n u \sin(n\theta + D).$$

This is the general form of the *first* integral of the proposed equation. In this equation let D be successively made equal to 0 and 90° , by which means we obtain two first integrals; and affixing the arbitrary constants implied in the sign of integration $\int_\theta$, they are,

$$\int_\theta \Theta \cos n\theta + C_1 = d_\theta u \cdot \cos n\theta + n u \cdot \sin n\theta,$$

$$\text{and } -\int_\theta \Theta \sin n\theta + C_2 = -d_\theta u \cdot \sin n\theta + n u \cdot \cos n\theta.$$

By multiplying these respectively by $\sin n\theta$ and $\cos n\theta$ and adding, we find

$$\sin n\theta \cdot \int_\theta \Theta \cos n\theta - \cos n\theta \int_\theta \Theta \sin n\theta + C_1 \sin n\theta + C_2 \cos n\theta = n u,$$

which may be put under the more simple form

$$u = A \cos(n\theta + B) + \frac{\sin n\theta}{n} \cdot \int_\theta \Theta \cos n\theta - \frac{\cos n\theta}{n} \cdot \int_\theta \Theta \sin n\theta,$$

A, B being arbitrary constants.

For the sake of simplicity let the motion be supposed to begin at the apse, in which case $C' = 0$, and

$$u = a' + C \cos n\theta,$$

$$\text{or } \frac{1}{r} = a' + C \cos n\theta;$$

which is the equation of the Moon's orbit.

140. Remark. The equation of an ellipse about the focus is

$$\frac{1}{r} = \frac{a}{b^2} \{1 + e \cos (\theta - \varpi)\},$$

ϖ being the distance of the apse from the prime radius measured in the same direction as θ . There is a remarkable similarity between this equation and that of the Moon's orbit, which will be rendered more evident by putting the latter under the form

$$\frac{1}{r} = \frac{a}{b^2} \{1 + e \cos (\theta - 1 - n \cdot \theta)\},$$

a , b and e being such that

$$\frac{a}{b^2} = a' = \frac{\mu}{h^2 - \mu'}, \quad \text{and } \frac{ae}{b^2} = C.$$

This we may consider as the equation of an ellipse, the angular distance of whose apse from the prime radius is $(1 - n) \theta$,

$$\text{or } \varpi = (1 - n) \theta;$$

$$\therefore \frac{d_t \varpi}{d_t \theta} = 1 - n = 1 - \left(1 - \frac{\mu'}{h^2}\right)^{\frac{1}{2}},$$

that is, the angular motion of the apse bears a constant ratio to that of the Moon.

Hence the Moon may be considered as moving in an ellipse, while the ellipse itself revolves in the same direction, with an angular velocity which bears a constant ratio to the angular velocity of the Moon.

141. The Moon will be at an apse whenever $d_\theta u = 0$, (Art. 110); but $d_\theta u = -nC \sin n\theta$; and consequently at an apse $n\theta = \pi, 2\pi, 3\pi, \dots$, the corresponding values of θ being

$$\theta = \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n} \dots$$

Hence the apsidal angle, or angle between two consecutive apses, is $\frac{\pi}{n} = \frac{\pi}{\left(1 - \frac{\mu}{h^2}\right)^{\frac{1}{2}}}$.

142. PROB. *A body revolves in an orbit, which is very nearly circular, acted on by a force varying as any function of the distance; to find the angle between the apsides.*

Let the force be F , and assume $\phi u = \frac{F}{u^3}$, and let c be the least value of u ; make $u = c + x$, x being by the question very small; then

$$\begin{aligned} F &= u^3 \phi(c + x), \\ &= u^3 (\phi c + \phi' c \cdot x + \dots) \\ &= u^3 \phi c + u^3 \phi' c \cdot x, \text{ nearly.} \end{aligned}$$

Also, because the orbit is very nearly circular, the velocity in the orbit is very nearly = velocity in a circle at the same distance; if therefore we assume (vel.)² at the least distance = $(1 - m) \cdot (\text{vel.})^2$ in a circle of radius $\frac{1}{c} = (1 - m) c \phi c$, m will be very small;

$$\therefore (1 - m) c \phi c = (\text{vel.})^2 = h^2 c^3;$$

$$\therefore h^2 = (1 - m) \frac{\phi c}{c}.$$

$$\text{But } d_\theta^2 u + u = \frac{\phi c}{h^2} + \frac{\phi' c}{h^2} x \dots \text{ Art. 107,}$$

$$Xdx + Ydy + Zdz$$

$$= \frac{c}{\phi c} (1 + m) (\phi c + \phi' c \cdot x) \text{ nearly,}$$

$$= (1 + m) c + \frac{c\phi' c}{\phi c} \cdot x \text{ nearly;}$$

$$\therefore d_0^2 u + \left(1 - \frac{c\phi' c}{\phi c}\right) u = (1 + m) c - \frac{c^2 \phi' c}{\phi c}, \because x = u - c.$$

This equation is of precisely the same form as that in Art. 139; consequently the apsidal angle

$$= \frac{\pi}{n} = \frac{\pi}{\left(1 - \frac{c\phi' c}{\phi c}\right)^{\frac{1}{2}}}.$$

MOTION OF A PARTICLE IN SPACE OF THREE DIMENSIONS.

143. *To investigate differential equations for the motion of a free particle when acted on by any accelerating forces.*

Refer the place of the particle at any time t to fixed co-ordinate axes, and let xyz be its co-ordinates. Resolve the accelerating forces which act upon it into sets parallel to the axes; denote them by XYZ . Then as in Art. 92, we have

$d_t x$ = velocity parallel to the axis of x ,

$d_t y$ = ... y ,

$d_t z$ = ... z .

Also $d_t^2 x = X$, $d_t^2 y = Y$, $d_t^2 z = Z$.

144. **PROP.** *When the centres of force are fixed points, XYZ are in nature always such as to make $Xdx + Ydy + Zdz$ integrable per se.*

For, all forces in nature are found to depend only upon the *distances* of the particle from the centres of force

Let a_1, b_1, c_1 be the co-ordinates of any centre of force, r_1 the distance of the particle from this centre, and $\phi_1(r_1)$ the accelerating force exerted by it upon the particle. Let

$a_2, b_2, c_2, r_2, \phi_2(r_2)$, &c. be similar quantities for other centres of force.

$$\text{Then } X = -\phi_1(r_1) \cdot \frac{x - a_1}{r_1} - \phi_2(r_2) \cdot \frac{x - a_2}{r_2} - \dots$$

$$\text{Now } r_1^2 = (x - a_1)^2 + (y - b_1)^2 + (z - c_1)^2;$$

$$\therefore r_1 d_x r_1 = x - a_1.$$

$$\text{Similarly } r_2 d_x r_2 = x - a_2,$$

$$\&c. = \&c,$$

$$\therefore X = -\phi_1(r_1) \cdot d_x r_1 - \phi_2(r_2) \cdot d_x r_2 - \dots$$

$$\therefore X dx = -\phi_1(r_1) \cdot d_x r_1 dx - \phi_2(r_2) \cdot d_x r_2 dx - \dots$$

$$\text{Similarly } Y dy = -\phi_1(r_1) \cdot d_y r_1 dy - \phi_2(r_2) \cdot d_y r_2 dy - \dots$$

$$\text{add } Z dz = -\phi_1(r_1) \cdot d_z r_1 dz - \phi_2(r_2) \cdot d_z r_2 dz - \dots$$

Adding these three equations together, and observing that

$$dr_1 = d_x r_1 dx + d_y r_1 dy + d_z r_1 dz, \&c.$$

we obtain

$$X dx + Y dy + Z dz = -\phi_1(r_1) dr_1 - \phi_2(r_2) dr_2 - \dots$$

Now $\phi_1(r_1) dr_1$ is a complete differential of some function of r_1 ; denote that function by $\psi_1 r_1$; and so of the other terms; then

$$\int (X dx + Y dy + Z dz) = -\psi_1(r_1) - \psi_2(r_2) - \dots$$

145. If the equations of motion in Art. 143 be multiplied respectively by $2 d_t x, 2 d_t y, 2 d_t z$; and the results be added together, we have

$$2 d_t x \cdot d_t^2 x + 2 d_t y \cdot d_t^2 y + 2 d_t z \cdot d_t^2 z = 2 X d_t x + 2 Y d_t y + 2 Z d_t z.$$

Now if v be the velocity of the body at the time t , since $d_t x, d_t y, d_t z$ are its velocities parallel to three rectangular axes;

$$\therefore v^2 = (d_t x)^2 + (d_t y)^2 + (d_t z)^2 \dots \dots \text{Art. 18, (5).}$$

By differentiating this equation we find

$$\begin{aligned} d_t(v^2) &= 2 d_t x d_t^2 x + 2 d_t y d_t^2 y + 2 d_t z d_t^2 z \\ &= 2 X d_t x + 2 Y d_t y + 2 Z d_t z, \end{aligned}$$

or according to the differential method of notation,

$$d(v^2) = 2Xdx + 2Ydy + 2Zdz;$$

$$\therefore v^2 = 2 \int (Xdx + Ydy + Zdz)$$

$$= -2\psi_1(r_1) - 2\psi_2(r_2) - \dots + \text{constant.}$$

If u be the velocity of the particle, at some known point, the distances of which from the centres of force are $R_1, R_2, \dots$ we have

$$u^2 = -2\psi_1(R_1) - 2\psi_2(R_2) - \dots + \text{constant.}$$

$$\therefore v^2 - u^2 = 2\psi_1(R_1) + 2\psi_2(R_2) + \dots$$

$$- 2\psi_1(r_1) - 2\psi_2(r_2) - \dots$$

This equation gives the velocity of the body at any proposed point of its path. It will be remarked however that in obtaining it we have made no hypothesis respecting the precise form of the path. The result is therefore true, whatever be the form of the path: and as this shews that v depends only on the values of $r_1, r_2, \dots$ the velocity which the body has at any given point will be independent of the course which it took to arrive there.

In Art. 102, we have considered the path of the body to lie in one plane, in which also the forces are supposed to act. We shall here investigate corresponding results when the forces do not act in a plane, and the path in consequence is a curve of double curvature.

146. *To investigate the differential equations of motion of a particle, when the forces acting on it are referred to the tangent line and normal plane of the path at the point where the particle is at any moment.*

Let R, S, T be the resolved forces; S being in direction of a tangent, and R, T acting along two lines at right angles to each other in the normal plane. Also, let θ, ϕ, ψ be the inclinations of R to fixed rectangular axes of x, y, z : and $\theta_1, \phi_1, \psi_1, \theta_2, \phi_2, \psi_2$ similar quantities for S, T . Then resolving R, S, T parallel to x, y, z , we have

$$\left. \begin{aligned} d_i^2 x &= R \cos \theta + S \cos \theta_1 + T \cos \theta_2 \\ d_i^2 y &= R \cos \phi + S \cos \phi_1 + T \cos \phi_2 \\ d_i^2 z &= R \cos \psi + S \cos \psi_1 + T \cos \psi_2 \end{aligned} \right\} \dots\dots\dots (A).$$

Multiply these equations by $\cos \theta_1, \cos \phi_1, \cos \psi_1$ respectively, and add the results, (observing that, because R, S, T act at right angles to each other,

$$\cos \theta \cos \theta_1 + \cos \phi \cos \phi_1 + \cos \psi \cos \psi_1 = 0,$$

$$\text{and } \cos \theta_2 \cos \theta_1 + \cos \phi_2 \cos \phi_1 + \cos \psi_2 \cos \psi_1 = 0);$$

$$\therefore \cos \theta_1 d_i^2 x + \cos \phi_1 d_i^2 y + \cos \psi_1 d_i^2 z = S.$$

But S acts along the tangent, and therefore

$$\cos \theta_1 = d_i x = \frac{d_i x}{d_i s}, \quad \cos \phi_1 = \frac{d_i y}{d_i s}, \quad \cos \psi_1 = \frac{d_i z}{d_i s},$$

and consequently

$$d_i x d_i^2 x + d_i y d_i^2 y + d_i z d_i^2 z = S d_i s,$$

$$\text{or} \quad d_i s d_i^2 s = S d_i s;$$

$$\therefore S = d_i^2 s \dots\dots\dots (1),$$

which agrees with equation (1) of Art. 102.

Again, multiply the equations (A) respectively by $\cos \theta, \cos \phi, \cos \psi$, and add the results, omitting the terms which vanish on the same principle as before; then

$$\cos \theta d_i^2 x + \cos \phi d_i^2 y + \cos \psi d_i^2 z = R.$$

$$\text{Similarly, } \cos \theta_2 d_i^2 x + \cos \phi_2 d_i^2 y + \cos \psi_2 d_i^2 z = T.$$

Now the directions of R and T in the normal plane are as yet arbitrary; and therefore we are at liberty to assign one of them such a position as we may think proper: let then the direction of R be assumed to pass through the centre of absolute curvature; and, putting ρ for the radius of absolute curvature, it is known that

$$\frac{(d_i s)^2}{\rho^2} = (d_i^2 x)^2 + (d_i^2 y)^2 + (d_i^2 z)^2 - (d_i^2 s)^2,$$

$$\cos \theta = \frac{\rho}{(d_t s)^2} (d_t s d_t^2 x - d_t x d_t^2 s),$$

$$\cos \phi = \frac{\rho}{(d_t s)^2} (d_t s d_t^2 y - d_t y d_t^2 s),$$

$$\text{and } \cos \psi = \frac{\rho}{(d_t s)^2} (d_t s d_t^2 z - d_t z d_t^2 s).$$

The last three being written in the expression for R (remembering that $d_t x d_t^2 x + d_t y d_t^2 y + d_t z d_t^2 z = d_t s d_t^2 s$), give

$$\begin{aligned} R &= \frac{\rho}{(d_t s)^2} \{ (d_t^2 x)^2 + (d_t^2 y)^2 + (d_t^2 z)^2 - (d_t^2 s)^2 \} \\ &= \frac{(d_t s)^2}{\rho} \dots\dots\dots (2), \end{aligned}$$

which agrees with the equation in Art. 102.

To obtain T we must proceed as follows. Since both R and S act in the osculating plane, T acts at right angles to that plane, that is, it acts parallel to a line whose equations are

$$x, (d_t y d_t^2 x - d_t x d_t^2 y) + z, (d_t y d_t^2 z - d_t z d_t^2 y) = 0,$$

$$\text{and } y, (d_t x d_t^2 y - d_t y d_t^2 x) + z, (d_t x d_t^2 z - d_t z d_t^2 x) = 0.$$

Now represent

$$(d_t y d_t^2 z - d_t z d_t^2 y)^2 + (d_t z d_t^2 x - d_t x d_t^2 z)^2 + (d_t x d_t^2 y - d_t y d_t^2 x)^2$$

by D^2 ,

$$\text{and therefore } \cos \theta_2 = (d_t y d_t^2 z - d_t z d_t^2 y) D,$$

$$\cos \phi_2 = (d_t z d_t^2 x - d_t x d_t^2 z) D,$$

$$\text{and } \cos \psi_2 = (d_t x d_t^2 y - d_t y d_t^2 x) D.$$

Writing these values of $\cos \theta_2$, $\cos \phi_2$, $\cos \psi_2$ in the expression for T , we find $T = 0$.

Wherefore, the resultant of all the forces which act on a free particle lies in the osculating plane of its path; and the

centrifugal force tends from the centre of absolute curvature, and is equal to

$$\frac{(\text{velocity})^2}{\text{rad. of absolute curvature}}.$$

Remark. These results might have been deduced at once from Art. 102. For when the path of a particle is a curve of double curvature, three consecutive points lie in a plane, (the osculating plane). Now two consecutive points determine the velocity, and direction of motion; the third determines the *change* of velocity and of direction,—so that three consecutive points determine both the magnitude and direction of the deflecting force, and the magnitude of the tangential force. For three points then (which we have just shewn are sufficient to decide what are the forces which regulate the form of the path) the body may be considered as moving in a *plane curve*, and consequently the property of the forces proved in Art. 102 for plane curves applies to curves of double curvature, considered as coinciding at any point with the osculating plane.

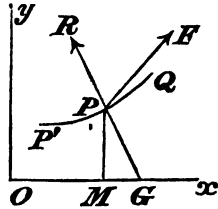
CHAPTER V.

ON THE CONSTRAINED MOTION OF A PARTICLE.

IN considering the motion of a particle subject to some restraint, such as the tension of a string or rod, the reaction of a curve line or of a surface; if we reckon the tensions and reactions among the impressed forces we may consider it as free, and apply the equations investigated in the last Chapter.

147. *PROB. A particle P moves upon a smooth plane curve PQ, being acted on by forces in the plane of the curve: to determine the velocity and pressure at any point.*

Resolve the accelerating forces which act upon the particle parallel to Ox , Oy , and call these components XY respectively. At the time t let P be the place of the particle, xy its co-ordinates OM , MP ; s equal $P'P$ the arc described from some fixed point P' in the curve. θ = inclination of the tangent at P to the axis of x ,



$$\therefore \tan \theta = d_x y.$$

Let m = mass of the particle, and draw GPR a normal at P .

Now because the curve is smooth its action is wholly in the direction of the normal PR ; denote the pressure which it exercises upon the particle by R . This is equivalent to an accelerating force $\frac{R}{m}$ (Art. 23) acting at P in the direction PR . Resolving this force parallel to Ox , Oy , we have the following equations of motion,

$$d_t^2 x = X - \frac{R}{m} \sin \theta,$$

$$d_t^2 y = Y + \frac{R}{m} \cos \theta.$$

Now $\sin \theta = d_t y / d_t s$, and $\cos \theta = d_t x / d_t s$; and hence by substituting these values in the above equations, we have

$$d_t^2 x = X - \frac{R}{m} \frac{d_t y}{d_t s} \dots\dots(1),$$

$$d_t^2 y = Y + \frac{R}{m} \frac{d_t x}{d_t s} \dots\dots(2).$$

Eliminating R , by multiplying the former by $2d_t x$, the latter by $2d_t y$ and adding the results;

$$\therefore 2d_t x d_t^2 x + 2d_t y d_t^2 y = 2X d_t x + 2Y d_t y.$$

Hence by integration,

$$(d_t x)^2 + (d_t y)^2 = 2 \int (X d_t x + Y d_t y) + C,$$

$$\text{or } v^2 = 2 \int (X dx + Y dy) + C \dots\dots\dots (3).$$

Again, to find R , multiply (1) by $d_t y$ and (2) by $d_t x$; the difference of the results is

$$d_t x d_t^2 y - d_t y d_t^2 x = Y d_t x - X d_t y + \frac{R}{m} d_t s.$$

But ρ being the radius of curvature at the point P , we have, as explained in Art. 102,

$$\frac{1}{\rho} = \frac{d_t x d_t^2 y - d_t y d_t^2 x}{(d_t s)^3};$$

$$\therefore \frac{(d_t s)^3}{\rho} = Y d_t x - X d_t y + \frac{R}{m} d_t s;$$

$$\therefore \frac{R}{m} = X \frac{d_t y}{d_t s} - Y \frac{d_t x}{d_t s} + \frac{(d_t s)^2}{\rho};$$

$$\therefore R = m X d_t y - m Y d_t x + \frac{m v^2}{\rho} \dots\dots(4).$$

This expression consists of two distinct parts, of which the latter $\frac{mv^2}{\rho}$ is that portion of R which is employed in counter-acting the centrifugal force of the particle (Art. 102).

The other portion

$X d_y y - Y d_x x = X \sin \theta - Y \cos \theta = X \cos PGO - Y \sin PGO$ is equal to the impressed accelerating force in the direction of the normal PG , and therefore $m(X d_y y - Y d_x x)$ = the pressure of m by the action of the impressed forces. Consequently the whole pressure R

$$= \begin{cases} \text{statical pressure due to the impressed forces} \\ \pm \text{statical pressure due to centrifugal force:} \end{cases}$$

the sign + or - is to be used according as the centrifugal force increases or diminishes the pressure on the curve line; *i. e.* according as the particle moves on the concave or convex side of the curve.

148. Since $X dx + Y dy$ is an exact differential of a function of x and y , (Art. 144) the value of its integral between given limits will be independent of the particular relation between x and y which constitutes the equation of the curve line along which the body moves. Hence in moving from one given position to another the velocity acquired is independent of the form of the curve on which the particle is constrained to move.

149. When a body moves on the convex side of a curve, the centrifugal force may become so great as to cause the body to fly off; and when it moves on the concave side, it may become too small to keep the body against the curve. The point where the body, in either case, will quit the curve, may be found by putting the expression for R equal to 0. Taking the case represented by the figure, we have

$$\begin{aligned} v^2 &= Y \rho d_x x - X \rho d_y y \\ &= Y \cdot \rho \sin PGx + X \cdot \rho \cos PGx. \end{aligned}$$

Let now F be the resultant of the impressed forces, and let θ' be the inclination of PF to the axis of x ;

$$\therefore X = F \cos \theta', \quad Y = F \sin \theta';$$

$$\begin{aligned} \therefore v^2 &= F \sin \theta' \rho \sin PGx + F \cos \theta' \rho \cos PGx \\ &= F \rho \cos (PGx - \theta') \\ &= F \rho \cos RPF. \end{aligned}$$

Hence if χ be the chord of curvature drawn from P in the direction PF ,

$$\begin{aligned} v^2 &= \frac{1}{2} F \chi \\ &= 2F \cdot \frac{\chi}{4}. \end{aligned}$$

Consequently the body flies off at that point where its velocity is that due to one fourth of the chord of curvature. (Compare this result with Art. 117.)

150. As an example of the application of Art. 147, let us take the following.

PROB. *If by means of a string a heavy body be whirled round in a vertical circle, with the least possible velocity, the tension of the string at the lowest point will be six times the weight of the body.*

Let a = the radius of the circle or length of the string,

m = the mass of the body,

u = the velocity at the highest point of the circle.

The least possible velocity is that which just suffices to keep the string stretched when the body is at the highest point of the circle; at that point therefore the tension, or value of R , must be equal to 0;

$$\therefore 0 = \text{statical weight} - \text{centrifugal weight}$$

$$= mg - \frac{mu^2}{a};$$

$$\therefore u^2 = ag.$$

$$\begin{aligned}
 \text{But in general } v^2 &= 2 \int (Xdx + Ydy) + C \\
 &= 2 \int (gdy) + C \text{ in this case} \\
 &= 2gy + C,
 \end{aligned}$$

y being measured from the highest point in the direction in which g acts.

$$\text{When } y = 0, v^2 = u^2 = ag; \therefore C = ag;$$

$$\therefore v^2 = 2gy + ag,$$

at the lowest point $y = 2a$; hence at that point

$$(\text{vel.})^2 = 4ag + ag = 5ag,$$

and the pressure = statical weight + centrifugal weight

$$\begin{aligned}
 &= mg + m \cdot \frac{(\text{vel.})^2}{a} \\
 &= mg + m \cdot \frac{5ag}{a} \\
 &= 6 \cdot mg = 6 \cdot \text{weight of } m.
 \end{aligned}$$

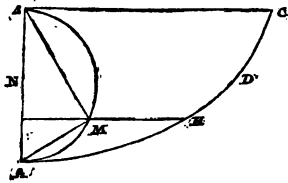
151. PROB. *To find the time of descent down the arc of a cycloid, whose axis is vertical, and base horizontal.*

Let D be the point from which the descent commences, and E the place of the body after an interval t . Draw EMN horizontal, cutting the circle AMB in M : join AM , MB . Then if BA be taken to represent the force of gravity, MA which is parallel to a tangent at E , will represent the force which urges the body down the curve; and the part BM being perpendicular to the curve, combines with the centrifugal force, and produces pressure on the curve, but in no wise affects the motion.

Hence,

$$\frac{\text{accelerating force down the curve}}{g} = \frac{MA}{AB} = \frac{2MA}{2AB} = \frac{AE}{AC}$$

Let now $AE = s$, $AC = l$, $AD = k$,



then the force down the curve $= \frac{s}{l} \cdot g$;

$$\therefore -d_t^2 s = \frac{s}{l} \cdot g \dots \dots \text{Art. 102.}$$

Multiply by $-2d_t s$, and integrate;

$$\therefore (d_t s)^2 = \frac{g}{l} (-s^2 + C).$$

But at D , $s = k$, and the velocity $d_t s = 0$; $\therefore C = k^2$;

$$\therefore (d_t s)^2 = \frac{g}{l} (k^2 - s^2);$$

$$\therefore -d_t s = \left(\frac{g}{l}\right)^{\frac{1}{2}} \cdot \sqrt{k^2 - s^2} \dots \dots \text{Art. 83};$$

$$\therefore \left(\frac{g}{l}\right)^{\frac{1}{2}} = \frac{-d_t s}{\sqrt{k^2 - s^2}};$$

$$\therefore t \left(\frac{g}{l}\right)^{\frac{1}{2}} = \cos^{-1} \frac{s}{k} + C.$$

But at D , $t = 0$, $s = k$, and therefore $C = 0$;

$$\therefore t = \left(\frac{l}{g}\right)^{\frac{1}{2}} \cdot \cos^{-1} \frac{s}{k}.$$

At the expiration of the descent $s = 0$;

$$\begin{aligned} \therefore \text{the time of descent} &= \left(\frac{l}{g}\right)^{\frac{1}{2}} \cdot \cos^{-1} 0 \\ &= \frac{\pi}{2} \left(\frac{l}{g}\right)^{\frac{1}{2}}. \end{aligned}$$

See Art. 73, where the same result is deduced by another process. On account of the time of descent being the same whether the body descend through a longer or a shorter arc, the cycloid is said to be *tautochronous*.

152. The equation of motion in the preceding article is of a form of such frequent occurrence as to be worthy of particular attention: it is sometimes called the 'equation of

vibratory motion:’ and it will be found useful to remember that whenever an equation of motion takes the form

$$d_t^2 s + n^2 s = a,$$

the motion represented by it is one of tautochronous vibration: the time of one vibration, that is, the time which elapses between s having its extreme values, being independent of those limiting values, and equal to $\frac{\pi}{n}$.

153. It is the property of tautochronism which has rendered it expedient to cause the pendulums of clocks to vibrate in cycloids. But not much advantage is gained over the common arrangement unless the impulses by which the pendulum is kept in a vibrating state be communicated to the pendulum at a proper instant. We may know when this is, from the consideration, that each half of the cycloid being tautochronous, a change of velocity at the lowest point of the cycloid, though it may alter the length of the *arc* of ascent will have no effect upon the *time*: while a change of velocity at any other point of the arc will cause the remainder of the arc to be described with too great or too small a velocity, and the time of motion will be affected. To make vibrations cycloidal a pendulum is suspended from its point of support by means of a short elastic lamina of steel, the thickness of which is so regulated as to accomplish the required purpose. The impulse on the pendulum is communicated at the lowest point of the vibration.

We shall now shew that if the pendulum is subject to a constant friction, or uniform retarding force, the time of vibration is not thereby affected.

154. *PROB. To find the effect of a uniform retarding force upon the time and arc of vibration of a body describing a cycloid.*

Using the same notation as before, putting f for the constant retarding force, and $n^2 = \frac{g}{l}$ for brevity, we have for the motion of the body,

$$-d_t^2 s = n^2 s - f$$

which may easily be put under the form

$$d_t^2 \left(s - \frac{f}{n^2} \right) + n^2 \left(s - \frac{f}{n^2} \right) = 0.$$

This being of the form pointed out in Art. 152, the time of vibration is $\frac{\pi}{n}$, and is therefore the same as when there is no friction.

If this equation be integrated once as in Art. 151, we find

$$(d_t s)^2 = n^2 \left\{ \left(k - \frac{f}{n^2} \right)^2 - \left(s - \frac{f}{n^2} \right)^2 \right\}.$$

Now at the beginning and end of a vibration $d_t s = 0$, and therefore

$$k - \frac{f}{n^2} = \pm \left(s - \frac{f}{n^2} \right).$$

The two values of s derived from this equation are

$$s = k, \text{ and } s = - \left(k - \frac{2f}{n^2} \right);$$

the former of which represents any arc of descent, and the latter the next arc of ascent: consequently $\frac{2f}{n^2}$ the decrement of the arc of ascent at each oscillation is constant. It is to obviate this decrement that the blow on the pendulum by the pallet wheel of a clock is necessary.

155. If besides friction, the pendulum should be affected by another force, which varies as s , the arc from the lowest point, such a force would be taken account of by altering the value of n . Consequently though the absolute time of vibration would be altered, the property of tautochronism and therefore the practical value of the pendulum would remain unaffected.

156. PROB. *To find the effect of a small force, which varies as the velocity, upon the time and arc of vibration.*

In this case the equation of motion is

$$-d_t^2 s = n^2 s + 2\mu d_t s$$

μ being + or - according as the force assists or hinders the action of gravity down the arc.

We assume that μ is so small as to allow the roots of the equation $x^2 + 2\mu x + n^2 = 0$ to be impossible. If we write n' for $\sqrt{n^2 - \mu^2}$, these roots are $-\mu \pm n' \sqrt{-1}$, and the integral of the equation of motion is

$$s = A e^{-\mu t} \cos (n' t + B).$$

$$\text{Hence } d_t s = -A e^{-\mu t} \{n' \sin (n' t + B) + \mu \cos (n' t + B)\}.$$

Now at the commencement of a vibration $s = k$, $d_t s = 0$, $t = 0$,

$$\therefore k = A \cos B; \text{ and } 0 = n' \sin B + \mu \cos B,$$

$$\therefore B = -\tan^{-1} \frac{\mu}{n'},$$

$$\text{and } A = \frac{k \sqrt{\mu^2 + n'^2}}{n'} = \frac{n k}{n'};$$

$$\therefore s = \frac{n k}{n'} \cdot e^{-\mu t} \cos \left(n' t - \tan^{-1} \frac{\mu}{n'} \right),$$

$$\text{and } d_t s = -\frac{n k}{n'} e^{-\mu t} \{n' \sin (n' t + B) + \mu \cos (n' t + B)\},$$

which, by expanding the terms $\sin (n' t + B)$, $\cos (n' t + B)$, becomes

$$\begin{aligned} d_t s &= -\frac{n k}{n'} e^{-\mu t} (n' \cos B - \mu \sin B) \sin n' t, \\ &= -\frac{n^2 k}{n'} \cdot e^{-\mu t} \sin n' t. \end{aligned}$$

Now at the beginning and end of a vibration $d_t s = 0$, and therefore

$$n' t = 0, \text{ or } \pi, \text{ or } 2\pi, \dots$$

Consequently the time of a vibration

$$= \frac{\pi}{n'} = \frac{\pi}{\left(\frac{g}{l} - \mu^2\right)^{\frac{1}{2}}},$$

which being independent of k shews that the property of the tautochronism of the cycloid remains in this case also: but the form of the denominator of the above fraction shews that the time of vibration is enlarged, and that, whether the disturbing force assist or hinder gravity, for μ enters the result only with an even power.

If k' denote the arc of ascent, which the body describes after falling down k , then $s = k'$ when $t = \frac{\pi}{n'}$, and therefore

$$\begin{aligned} k' &= \frac{nk}{n'} e^{-\frac{\mu\pi}{n'}} \cos \left(\pi - \tan^{-1} \frac{\mu}{n'} \right) \\ &= -k e^{-\frac{\mu\pi}{n'}}. \end{aligned}$$

Hence the successive arcs of ascent form a geometric progression whose common ratio is $e^{-\frac{\mu\pi}{n'}}$.

We shall now give the solution of the problem in its more general form, in a manner which has the advantage of greater simplicity than any which has hitherto been published.

157. **PROB.** *A body vibrates in a cycloidal arc, to find the effect of any small tangential disturbing force upon the time and arc of vibration.*

Represent, as before, the disturbing force by f , which we shall account + or - according as it assists or hinders gravity. The equation of motion is

$$\begin{aligned} -d_t^2 s &= n^2 s + f, \\ \text{or } d_t^2 s + n^2 s &= -f. \end{aligned}$$

Multiply by $\cos nt$ and integrate by parts (as in the note to Art. 139),

$$\therefore d_t s \cdot \cos nt + ns \sin nt = C_1 - \int f \cos nt \, dt \dots \dots (1).$$

Similarly, using $\sin nt$ for $\cos nt$, we have

$$d_t s \cdot \sin nt - ns \cos nt = C_2 - \int f \sin nt \, dt \dots \dots (2).$$

These are the two *first* integrals of the equation of motion. Now when the vibration is completed let the value of t be τ , and that of s be k' ; then taking the above integrals between $t = 0$ and $t = \tau$; or $s = k$ and $s = k'$, when $d_t s = 0$; we have

$$nk' \sin n\tau = \int_0^\tau f \cos nt \, dt \quad \text{from (1),}$$

$$\text{and } -nk' \cos n\tau = -nk + \int_0^\tau f \sin nt \, dt \quad \text{from (2).}$$

Eliminating k' between these we find

$$nk \sin n\tau = \int_0^\tau f \cos n(t - \tau) \, dt \dots (3),$$

and multiplying the same equations by $\sin n\tau$, $\cos n\tau$ respectively, and taking the difference of the results we have

$$nk' = nk \cos n\tau - \int_0^\tau f \sin n(t - \tau) \, dt \dots (4).$$

If f be given in terms of t these equations being integrated will give what is required: (3) will give the value of τ , the time of one vibration; and (4) will then give the value of k' , the arc of ascent.

158. If the value of f depend upon k , and equat. (3) be expanded (if necessary) and arranged in powers of k , the condition of tautochronism will be that the equation must be satisfied for all values of k , and therefore the coefficients of the different powers of k must be separately equal to zero.

From this we learn at once, that if f do not depend upon the *first* power of k , there cannot be tautochronism unless

$\sin n\pi = 0$, or $\tau = \frac{\pi}{n}$; that is, the time of vibration is the

same as when there is *no* disturbing force. Art. 154 is an example of this kind; and Arts. 155, 156 are instances of this not being the case when f does depend upon the first power of k .

159. When f is not given in terms of t we must be content with an approximate solution, if the equation of motion cannot be integrated directly. The approximation may be conducted on the principle, that since f is small, higher powers of f than the first may be omitted. Now $n\tau$ differs from π only by reason of f , and therefore in the right-

hand members of (3) and (4) we may write π for $n\tau$, and use $\frac{\pi}{n}$ for τ as one limit of the integration. Then we shall have

$$\left. \begin{aligned} nk \sin n\tau &= \int_0^{\frac{\pi}{n}} f \cos nt \, dt \\ nk' &= nk \cos n\tau - \int_0^{\frac{\pi}{n}} f \sin nt \, dt \end{aligned} \right\} \dots\dots\dots (A).$$

Ex. Let f vary as some power of the velocity $= a(d_1s)^m$.

We must use an approximate value of d_1s , obtained from (1) and (2), on the supposition that $f = 0$, or from

$$\begin{aligned} d_1s \cdot \cos nt + ns \sin nt &= 0, \\ \text{and } d_1s \cdot \sin nt - ns \cos nt &= nk; \\ \therefore d_1s &= nk \sin nt, \\ \text{and } \therefore f &= \mu n^m k^m \sin^m nt. \end{aligned}$$

This being written in the first of the equations (A) gives

$$\begin{aligned} nk \sin n\tau &= \mu n^m k^m \int_0^{\frac{\pi}{n}} \sin^m nt \cos nt \, dt = 0; \\ \therefore \tau &= \frac{\pi}{n}. \end{aligned}$$

This result shews that the time of a complete oscillation is unaffected by the first power of the disturbing force.

The second of equations (A) becomes

$$\begin{aligned} nk' &= -nk - \mu n^m k^m \int_0^{\frac{\pi}{n}} \sin^{m+1} nt \, dt, \\ &= -nk + \mu n^{m-1} k^m \cdot \frac{m \cdot (m-2) \dots 3 \cdot 1}{(m+1)(m-1) \dots 4 \cdot 2} \pi, \quad (m \text{ odd}), \end{aligned}$$

or,

$$= -nk + 2\mu n^{m-1} k^m \cdot \frac{m \cdot (m-2) \dots 4 \cdot 2}{(m+1)(m-1) \dots 5 \cdot 3}, \quad (m \text{ even}).$$

When $m = 2$, which is the case for air nearly,

$$k' = -k + 2\mu k^3 \cdot \frac{2}{3},$$

$$\text{or, } -k' = k - \frac{4}{3}\mu k^3,$$

the last term of which is the decrement of the semi-arc of vibration.

160. The reader may consult with advantage a paper on this subject by Mr. Airy, printed in the *Transactions of the Cambridge Philosophical Society*, Vol. III. The results obtained in that paper are derived from the equation of motion by the method of parameters. But it is proper to remark, that a result respecting the time of oscillation is not of much value: if it be true only to the first power of the disturbing force: for an error in the time of oscillation, being added at every vibration, will soon, by accumulation, become sensible. And indeed it is not very difficult to shew that, generally, if the disturbing force retard the vibration both in the descending and ascending arcs, the effect upon the time must be of the second order. We have an example of this in Art. 156, where we have found

$$\frac{\pi}{\left(\frac{g}{l} - \mu^2\right)^{\frac{1}{2}}} = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}} + \frac{\mu^2}{2} \pi \left(\frac{l}{g}\right)^{\frac{3}{2}} + \dots$$

for the time of an oscillation, which is greater than when there is no disturbing force by a quantity depending on μ^2 . And, again, if a neighbouring mountain exert a disturbing force μ upon a pendulum in a horizontal direction, the time of an oscillation is

$$\begin{aligned} & \pi \left(\frac{l}{\sqrt{g^2 + \mu^2}} \right)^{\frac{1}{2}} \dots \text{Art. 74,} \\ & = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}} - \pi \left(\frac{l}{g}\right)^{\frac{1}{2}} \frac{\mu^2}{4g^2} + \dots \end{aligned}$$

consequently the mountain increases the number of oscillations in a day by a quantity depending on μ^2 .

And in general if μ be the coefficient in the expression which determines the magnitude of f , so that $f = \mu \phi(t)$, and T be the time of an undisturbed oscillation; by expanding in a series we shall have the time of a disturbed *descent*

$$= \frac{1}{2} T + A\mu + B\mu^2 + \dots$$

A, B , being functions of h the arc of descent.

The same formula will give the arc of ascent, by writing $-\mu$ for μ and k' for k . Consequently the time of the *ascent*

$$= \frac{1}{2} T - \{A + d_k A \cdot (k' - k) + \dots\} \mu \\ + \{B + d_k B \cdot (k' - k) + \dots\} \mu^2 - \dots$$

Hence by addition, we find the whole time of an oscillation

$$= T - d_k A \cdot (k' - k) \mu + 2 B \mu^2 + \dots$$

But there is here no term of the *first* order in the part which follows T , for $(k' - k) \mu$ must be of the second, or a higher order. Consequently, in general the effect upon the time of oscillation will not be of the first order.

161. PROB. *To investigate the equation of the tautochronous curve, when the force which urges the body is not constant.*

Using the same figure and notation as in Art. 151, let AEC be one half of the curve required, A being its lowest point. Resolve the force which produces the motion into components, one parallel and the other perpendicular to the tangent at E ; and denote the former by $\frac{1}{2} \phi s$,

$$\therefore -v d_s v = \frac{1}{2} \phi s;$$

$$\therefore v^2 = -\int_s \phi s + C.$$

Let $\int_s \phi s = \psi s$; then at D , $v = 0$ and $s = k$, $\therefore C = \psi(k)$;

$$\therefore v^2 = \psi(k) - \psi(s);$$

$$\therefore -d_s s = \sqrt{\psi(k) - \psi(s)};$$

$$\text{and } \therefore t = \int_s^k \frac{-1}{\sqrt{\psi(k) - \psi(s)}},$$

the integral is to be taken between $s = k$ and $s = 0$, and when so taken the result is by the question to be independent of k .

Now expanding the denominator by the Binomial Theorem,

$$t = - \int \left\{ \frac{1}{(\psi k)^{\frac{1}{2}}} + \frac{1}{2} \cdot \frac{\psi s}{(\psi k)^{\frac{3}{2}}} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{(\psi s)^2}{(\psi k)^{\frac{5}{2}}} + \dots \right\}.$$

But the first term, taken between proper limits,

$$= \frac{k}{(\psi k)^{\frac{1}{2}}},$$

which is to be independent of k for all values whatever; we may therefore write s for k without altering its value,

$$\therefore \frac{s}{(\psi s)^{\frac{1}{2}}} = \text{constant};$$

$$\therefore \psi(s) \propto s^2 = Cs^2 \text{ suppose};$$

$$\therefore \psi(k) = Ck^2.$$

If these values of $\psi(s)$ and $\psi(k)$ be written in the other terms of the above series, they are found, after integration, to be independent of k ; we have therefore no further condition to satisfy;

$$\therefore v^2 = C(k^2 - s^2);$$

$$\begin{aligned} \therefore Xdx + Ydy &= vdv \\ &= -Csds, \end{aligned}$$

which is the differential equation of the curve required.

162. If the force act in parallel lines and be constant, denote it by f and let it be parallel to the axis of x ;

$$\therefore X = -f \text{ and } Y = 0;$$

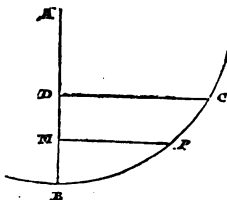
$$\therefore fdx = Csds;$$

$$\therefore 2fx = Cs^2,$$

which is the property of the cycloid: and hence tautochronism is peculiar to the cycloid when the force is constant.

163. PROB. To find the time of oscillation in a circular arc of given radius.

Let A be the centre, B the lowest point of the arc, C the point from which the body begins to fall; P the place of the body at any time t from leaving C . Draw CD , PM parallel to the horizon; and let $a = AB$, $h = BD$, $x = BM$, $s = BP$, and v = the velocity at P .



Then $X = -g$, and $Y = 0$;

$$\begin{aligned}\therefore v^2 &= 2 \int (X dx + Y dy) \dots\dots\dots \text{Art. 147 (3)} \\ &= 2 \int -g dx \\ &= -2gx + C.\end{aligned}$$

But at C , $v = 0$, and $x = h$, therefore $C = 2gh$;

$$\begin{aligned}\therefore v^2 &= 2g(h - x); \\ \therefore -d_1 s &= v = \sqrt{2g(h - x)}.\end{aligned}$$

But $y^2 = 2ax - x^2$,

$$\begin{aligned}\text{and therefore } d_1 s &= \sqrt{(d_1 x)^2 + (d_1 y)^2} \\ &= \frac{a d_1 x}{\sqrt{2ax - x^2}};\end{aligned}$$

$$\therefore -\frac{a d_1 x}{\sqrt{2ax - x^2}} = \sqrt{2g(h - x)};$$

$$\therefore \frac{\sqrt{2g}}{a} \cdot d_1 t = \frac{-1}{\sqrt{2ax - x^2} \cdot \sqrt{h - x}};$$

$$\therefore \frac{\sqrt{2g}}{a} \cdot t = \int \frac{-1}{\sqrt{hx - x^2} \cdot \sqrt{2a - x}} \dots (B).$$

The integral is to be taken between $x = h$, and $x = 0$ for the whole descent.

Now

$$\frac{1}{\sqrt{2a - x}} = \frac{1}{\sqrt{2a}} \cdot \left\{ 1 + \frac{1}{2} \cdot \frac{x}{2a} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^2}{4a^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^3}{8a^3} + \dots \right\};$$

$$\therefore 2t \left(\frac{g}{a} \right)^{\frac{1}{2}} = \int \frac{-1}{\sqrt{hx - x^2}} \left\{ 1 + \frac{1}{2} \cdot \frac{x}{2a} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^2}{4a^2} + \dots \right\}.$$

But by the *Integral Calculus*,

$$\int \frac{-x^2}{\sqrt{hx - x^2}} = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \cdot h^n \pi$$

between the proper limits, and

$$\int \frac{-1}{\sqrt{hx - x^2}} = \pi;$$

$$\therefore \int \left(\frac{-1}{\sqrt{hx - x^2}} \cdot \frac{x}{2a} \right) = \frac{1}{2} \cdot \frac{h\pi}{2a} \text{ (here } n=1 \text{)}$$

$$\int \left(\frac{-1}{\sqrt{hx - x^2}} \cdot \frac{x^2}{4a^2} \right) = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{h^2 \pi}{4a^2} \text{ (here } n=2 \text{);}$$

&c. = &c.;

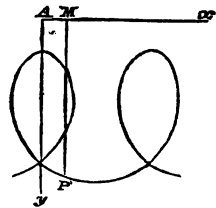
$$\therefore 2t \left(\frac{g}{a} \right)^{\frac{1}{2}} = \pi + \left(\frac{1}{2} \right)^2 \cdot \frac{h\pi}{2a} + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \cdot \frac{h^2 \pi}{4a^2} + \dots;$$

$$\therefore 2t = \pi \left(\frac{a}{g} \right)^{\frac{1}{2}} \cdot \left\{ 1 + \left(\frac{1}{2} \right)^2 \cdot \frac{h}{2a} + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \cdot \frac{h^2}{4a^2} + \dots \right\}.$$

Since t is the time of half an oscillation, $2t$ is the time of a whole oscillation. (The reader may compare this result with Art. 74.)

164. **PROB.** *A particle, acted on by gravity only, descends along a curve line in a vertical plane; to find the form of the curve that the same pressure may be exerted at every point.*

Let m be the mass of the particle, and suppose Ax the axis of x so taken that the velocity at any point P may be that due to MP , gravity acting in the direction Ay . Let $AM = x$, $MP = y$, s = the arc described in time t ; and let the pressure at any point equal k times the weight of the particle.



$k \cdot mg$ = pressure at P

= pressure due to gravity

+ pressure due to centrifugal force

$$= mgd_x + \frac{2mgy}{\text{rad. curvat.}}$$

But radius of curvature $= \frac{d_1 y}{d_1^2 x}$;

$$\therefore k = d_1 x + \frac{2 y d_1^2 x}{d_1 y}.$$

Multiply by $\frac{d_1 y}{2 \sqrt{y}}$;

$$\therefore \frac{k d_1 y}{2 \sqrt{y}} = \frac{d_1 x \cdot d_1 y}{2 \sqrt{y}} + \sqrt{y} \cdot d_1^2 x,$$

the integral of which is

$$k \sqrt{y} + C = \sqrt{y} \cdot d_1 x,$$

$$\text{or } d_1 x = k + \frac{C}{\sqrt{y}}.$$

As C is arbitrary we may write $\pm \sqrt{a}$ for it;

$$\therefore d_1 x = k \pm \left(\frac{a}{y}\right)^{\frac{1}{2}} \dots\dots (1).$$

Hence ρ the radius of curvature $= \frac{d_1 y}{d_1^2 x}$

$$= \mp \frac{2 y^{\frac{3}{2}}}{\sqrt{a}}.$$

Let θ = the inclination of the tangent at P to the axis of x , $\therefore \tan \theta = d_1 y$ and $\cos \theta = d_1 x$;

$$\therefore \cos \theta = k \pm \left(\frac{a}{y}\right)^{\frac{1}{2}} \dots\dots (2).$$

By means of this equation the curve may be traced; but the relation between x and y may be obtained, if necessary, from equation (1) by further integration.

1st. When $C = 0$, or $a = 0$, then $\cos \theta = k$, which is the equation of a straight line.

2nd. When k is positive and not < 1 .

In this case the upper sign of equation (2) cannot be used;

$$\therefore \cos \theta = k - \left(\frac{a}{y}\right)^{\frac{1}{2}}, \text{ and } \rho = \frac{2y^{\frac{1}{2}}}{\sqrt{a}}.$$

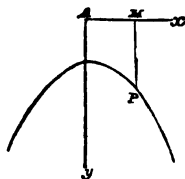
Hence the greatest value of y , or that which makes $\cos \theta = 1$, is $\frac{a}{(k-1)^2}$; and the least value, or that which makes $\cos \theta = -1$, is $\frac{a}{(k+1)^2}$; as y increases from the latter to the former, $\cos \theta$ passes through all its values; and this curve is always concave towards the axis of x , when $\cos \theta$ is positive, and convex when $\cos \theta$ is negative: for $\rho \cos \theta$ which is the projection of the radius of curvature upon the axis of $y = \frac{2y^{\frac{1}{2}}}{\sqrt{a}} \cdot \cos \theta$, which has the same sign as $\cos \theta$. Fig. p. 142 is the form of the curve of equal pressure in this case.

3rd. When k is positive and < 1 .

In this case both signs are allowable in equation (2), but the upper can apply only between those values of y which make $\cos \theta$ not > 1 , which

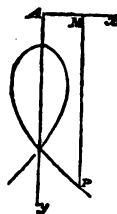
are $\frac{a}{(1-k)^2}$ and ∞ .

For this branch of the curve $\rho \cos \theta = -\frac{2y^{\frac{1}{2}}}{\sqrt{a}} \cdot \cos \theta$; and hence it is convex towards the axis of x .



The lower sign can apply only between those values of y which make the cosine fall between k and -1 ; these are $y = \infty$ and $y = \frac{a}{(k+1)^2}$.

As long as $\cos \theta$ is positive the curve is concave towards the axis of x , and the remaining part is convex.



165. PROP. *A particle is placed in a very narrow smooth tube in the form of a curve of double curvature; or we may suppose a very small ring through which passes a curve of double curvature; to determine the motion, the body being acted on by any forces.*

Let xyz be the co-ordinates of the particle m at the time t ; s the arc described; XYZ the forces impressed upon it; R the reaction of the tube, which, as the tube is smooth, must necessarily be in the normal plane; let α, β, γ be the angles which the direction of R makes with the co-ordinate axes. Then, since the angle between R , and the tangent to the curve at xyz is a right angle,

$$\therefore 0 = \cos \alpha \, d_t x + \cos \beta \, d_t y + \cos \gamma \, d_t z;$$

and therefore multiplying by $d_t s$

$$0 = \cos \alpha \cdot d_t x + \cos \beta \cdot d_t y + \cos \gamma \, d_t z \dots (1).$$

The equations of motion are

$$d_t^2 x = X + \frac{R}{m} \cdot \cos \alpha \dots\dots (2),$$

$$d_t^2 y = Y + \frac{R}{m} \cdot \cos \beta \dots\dots (3),$$

$$d_t^2 z = Z + \frac{R}{m} \cdot \cos \gamma \dots\dots (4).$$

Multiply these by $d_t x, d_t y, d_t z$ respectively, and add the results, taking account of (1),

$$d_t x d_t^2 x + d_t y d_t^2 y + d_t z d_t^2 z = X d_t x + Y d_t y + Z d_t z,$$

$$\text{or } d_t s d_t^2 s = X d_t x + Y d_t y + Z d_t z;$$

$$\therefore (d_t s)^2 = 2 \int (X dx + Y dy + Z dz) \dots (A).$$

This expression for the square of the velocity is independent of R ; and as the expression under the integral sign is integrable *per se* (Art. 144), *i.e.* without assuming any relation among the co-ordinates xyz , the value of the integral taken between given limits will be independent of the relation

between x, y, z at other points,—in other words,—independent of the form of the curve which joins the fixed limits. Hence in passing from one fixed point to another, the velocity acquired is entirely independent of the form of the curve line on which it passes from one point to the other.

Let now θ, ϕ, ψ denote the inclinations of the radius of absolute curvature to the axes of co-ordinates; and θ_2, ϕ_2, ψ_2 the inclinations of a line in the normal plane perpendicular to θ, ϕ, ψ . Then multiplying equations (2), (3), (4) by the cosines of θ, ϕ, ψ respectively, and adding the results, remembering that we have proved in Art. 146, that

$$\cos \theta d_i^2 x + \cos \phi d_i^2 y + \cos \psi d_i^2 z = \frac{(d_i s)^2}{\rho}, \text{ we find}$$

$$\begin{aligned} \frac{(d_i s)^2}{\rho} &= X \cos \theta + Y \cos \phi + Z \cos \psi \\ &\quad + \frac{R}{m} (\cos \alpha \cos \theta + \cos \beta \cos \phi + \cos \gamma \cos \psi). \end{aligned}$$

But the term $R (\cos \alpha \cos \theta + \cos \beta \cos \phi + \cos \gamma \cos \psi)$ is the resolved part of R in the direction of the radius of absolute curvature; if we call this R' , we have

$$R' = -m (X \cos \theta + Y \cos \phi + Z \cos \psi) + m \frac{(d_i s)^2}{\rho} \dots (B).$$

Again, if we multiply equations (2), (3), (4) by $\cos \theta_2, \cos \phi_2, \cos \psi_2$ and add, remembering that it has been proved in Art. 146 that $\cos \theta_2 d_i^2 x + \cos \phi_2 d_i^2 y + \cos \psi_2 d_i^2 z = 0$; and using T' to denote the resolved part of R in the direction θ_2, ϕ_2, ψ_2 , we find

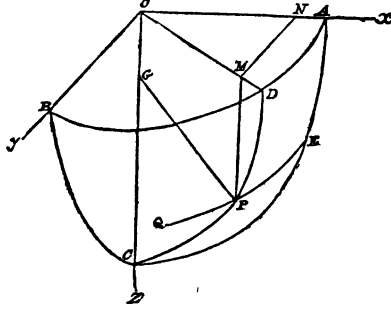
$$0 = X \cos \theta_2 + Y \cos \phi_2 + Z \cos \psi_2 + \frac{T'}{m};$$

$$\therefore T' = -m (X \cos \theta_2 + Y \cos \phi_2 + Z \cos \psi_2) \dots (C).$$

Equations (B), (C) give R' T' , the components of R , and shew that the whole pressure of the particle upon the curve is equal to the pressure due to the impressed forces + the pressure due to centrifugal force.

166. **PROB.** *A particle placed within a smooth surface of revolution is acted on by forces in the plane of a generating curve, i. e. in the plane passing through the particle and the axis : to determine the motion.*

Let OC the axis of revolution be that of x ; OA, OB those of x and y ; P the particle at the time t , when its co-ordinates are ON, NM, MP ; EP the path of the body; CPD the generating curve in which P is situated, and in the plane of which the forces act. Resolve the forces into two sets, Z parallel to OZ , and F parallel to OD : and let R



be the reaction of the surface in the direction of a normal PG . $ON = x$, $NM = y$, $MP = z$, $OM = r$, $\theta = DOA$, m = the mass of the particle. The accelerating force due to the pressure R is equivalent to $\frac{R}{m} \cos PGC$, and $-\frac{R}{m} \sin PGC$ in the directions OZ, OD . Hence we may consider the body as free and acted on by the two accelerating forces

$$Z - \frac{R}{m} \cos PGC \text{ parallel to } OZ,$$

$$\text{and } F - \frac{R}{m} \sin PGC \text{ parallel to } OD.$$

The latter may be resolved parallel to Ox and Oy ; and this being done, we have from the equations of free motion, (Art. 143.)

$$d_t^2 x = (F - \frac{R}{m} \sin PGC) \cos \theta,$$

$$d_t^2 y = (F - \frac{R}{m} \sin PGC) \sin \theta,$$

$$d_t^2 z = Z - \frac{R}{m} \cos PGC.$$

Now let the arc DP of the generating curve $= s$. Then because PG is perpendicular to the tangent of the generating curve at P ;

$$\therefore \cotan PGC = -d_r r;$$

$$\therefore \sin PGC = d_s s, \text{ and } \cos PGC = -d_r r.$$

It is evident also that $\cos \theta = \frac{x}{r}$ and $\sin \theta = \frac{y}{r}$.

$$\text{Hence, } d_t^2 x = (F - \frac{R}{m} d_s s) \cdot \frac{x}{r} \dots \dots \dots (1),$$

$$d_t^2 y = (F - \frac{R}{m} d_s s) \cdot \frac{y}{r} \dots \dots \dots (2),$$

$$d_t^2 z = Z + \frac{R}{m} d_r r \dots \dots \dots (3).$$

In using these equations, we are to remember that

$$r^2 = x^2 + y^2, \text{ and } (d_r r)^2 + (d_s s)^2 = 1.$$

167. PROB. *A particle acted on by any forces moves upon a curve surface; to determine the motion.*

Let Ox, Oy, Oz be the co-ordinate axes; AC, BC, ADB , sections of the given surface by the co-ordinate planes. DPC a section of it by a plane passing through Ox and the particle P at the time t from the beginning of the motion at E . EPQ the path of the body, the mass of which put $= m$. $ON = x, MN = y, MP = z, EP = s$; α, β, γ the inclinations of the normal PG (which does not now necessarily meet OC) to the axes OA, OB, OC ; R = the reaction of the surface, which takes place in that direction; this produces an accelerating force $= \frac{R}{m}$, which being resolved parallel to OA, OB, OC , gives

$$-\frac{R}{m} \cos \alpha \text{ parallel to } OA,$$

$$-\frac{R}{m} \cos \beta \text{ parallel to } OB,$$

$$-\frac{R}{m} \cos \gamma \text{ parallel to } OC.$$

Hence, if X, Y, Z be the impressed accelerating forces parallel to the same axes, the equations of motion are

$$d_t^2 x = X - \frac{R}{m} \cos \alpha \dots\dots\dots(1),$$

$$d_t^2 y = Y - \frac{R}{m} \cos \beta \dots\dots\dots(2),$$

$$d_t^2 z = Z - \frac{R}{m} \cos \gamma \dots\dots\dots(3).$$

But by the principles of Analytical Geometry,

$$\cos \alpha = \frac{-p}{\sqrt{1+p^2+q^2}}, \quad \cos \beta = \frac{-q}{\sqrt{1+p^2+q^2}},$$

$$\cos \gamma = \frac{1}{\sqrt{1+p^2+q^2}},$$

where p, q represent the partial differential coefficients $d_x z, d_y z$, derived from the equation to the given surface; they are connected by the equation

$$dz = p dx + q dy;$$

$$\therefore \cos \gamma \cdot dz = -\cos \alpha \cdot dx - \cos \beta \cdot dy,$$

$$\text{or } \cos \alpha \cdot dx + \cos \beta \cdot dy + \cos \gamma \cdot dz = 0.$$

To find v the velocity of the particle at P .

Multiply (1) by $2d_t x$, (2) by $2d_t y$, (3) by $2d_t z$ and add and integrate;

$$\therefore v^2 = (d_t x)^2 + (d_t y)^2 + (d_t z)^2$$

$$= 2 \int (X d_t x + Y d_t y + Z d_t z) + C,$$

$$\text{or } = 2 \int (X dx + Y dy + Z dz) + C \dots\dots(4).$$

This expression for v^2 is independent of R , and as the expression under the integral sign is integrable *per se*, i.e. without assuming any relation to exist between x, y, z , the integral, taken between given limits, will be dependent only upon the values of x, y, z at those limits, and not upon the general relation between x, y, z in the intermediate points of the path. Whatever therefore be the relation between x, y, z in general, the value of v will be the same as long as the limits of the path are fixed. Hence whatever be the form of the surface on which the body moves, the velocity which it acquires in passing thereon from one given point to another given point will be the same.

To find the path.

We must eliminate R from the differential equations of motion.

Multiply (3) by p and add (1), observing that

$$\cos \alpha + p \cos \gamma = 0;$$

$$\therefore \left. \begin{aligned} d_t^2 x + p d_t^2 z &= X + p Z \\ \text{similarly } d_t^2 y + q d_t^2 z &= Y + q Z \end{aligned} \right\} \dots\dots(B).$$

From these two equations we must endeavour either to eliminate t ; or to deduce from them one integrable equation, and then between this and (A) eliminate t . This cannot always be effected; when it can be done, the relation between x, y, z so found, and the given equation of the surface will be the equations of the path.

To find the pressure of the body upon the surface.

Multiply (1) by $\cos \alpha$, (2) by $\cos \beta$, (3) by $\cos \gamma$ and add the results, remembering that

$$1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma \quad (\text{Art. 19});$$

$$\therefore \cos \alpha \cdot d_t^2 x + \cos \beta \cdot d_t^2 y + \cos \gamma \cdot d_t^2 z$$

$$= X \cos \alpha + Y \cos \beta + Z \cos \gamma - \frac{R}{m}.$$

Substituting for $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ their values in terms of p and q .

$$\cos \alpha \cdot d_t^2 x + \cos \beta \cdot d_t^2 y + \cos \gamma \cdot d_t^2 z = \frac{d_t^2 z - p d_t^2 x - q d_t^2 y}{\sqrt{1 + p^2 + q^2}}.$$

$$\text{But } d_t z = p d_t x + q d_t y;$$

$$\therefore d_t^2 z = p d_t^2 x + q d_t^2 y + d_t p \cdot d_t x + d_t q \cdot d_t y.$$

$$\text{But } d_t p = d_x p \cdot d_t x + d_y p \cdot d_t y = d_x^2 z \cdot d_t x + d_x d_y z \cdot d_t y,$$

$$\text{similarly } d_t q = d_y^2 z \cdot d_t y + d_x d_y z \cdot d_t x;$$

$$\begin{aligned} \therefore d_t^2 z &= p d_t^2 x + q d_t^2 y + d_x^2 z \cdot (d_t x)^2 + 2 d_x d_y z \cdot d_t x d_t y \\ &\quad + d_y^2 z \cdot (d_t y)^2; \end{aligned}$$

$$\therefore \frac{R}{m} = X \cos \alpha + Y \cos \beta + Z \cos \gamma$$

$$- \frac{d_x^2 z \cdot (d_t x)^2 + 2 d_x d_y z \cdot d_t x d_t y + d_y^2 z \cdot (d_t y)^2}{\sqrt{1 + p^2 + q^2}}.$$

But if λ, μ be the angles which the direction of the body's motion at P makes with the axes of x and y ,

$$d_t x = v \cos \lambda \quad \text{and} \quad d_t y = v \cos \mu;$$

$$\therefore \frac{R}{m} = X \cos \alpha + Y \cos \beta + Z \cos \gamma$$

$$- \frac{d_x^2 z \cdot \cos^2 \lambda + 2 d_x d_y z \cdot \cos \lambda \cos \mu + d_y^2 z \cdot \cos^2 \mu}{\sqrt{1 + p^2 + q^2}} \cdot v^2.$$

But by Hymers' Analytical Geometry of Three Dimensions, 2nd Edition, Art. 217, Cor., the quantity into which v^2 is here multiplied is the reciprocal of the radius of curvature at P ;

$$\therefore R = m (X \cos \alpha + Y \cos \beta + Z \cos \gamma) + \frac{mv^2}{\rho};$$

therefore the pressure

= statical pressure due to the impressed forces

+ statical pressure due to centrifugal force.

The centrifugal force increases or diminishes the pressure according as the body moves on the concave or convex side of the curve surface, and the sign + or - must be used accordingly.

168. As an example, *Let the given surface be a hemispherical bowl, and let the body be acted on only by gravity in the direction OZ.*

Here $X = 0$, $Y = 0$, $Z = g$.

The equation of the surface is $x^2 + y^2 + z^2 = a^2$, from which we find

$$p = -\frac{x}{z}, \quad q = -\frac{y}{z};$$

$$\therefore \cos \alpha = \frac{x}{a}, \quad \cos \beta = \frac{y}{a}, \quad \cos \gamma = \frac{z}{a},$$

and the equations of motion become

$$d_t^2 x = -\frac{R}{m} \cdot \frac{x}{a}, \quad d_t^2 y = -\frac{R}{m} \cdot \frac{y}{a}, \quad d_t^2 z = g - \frac{R}{m} \cdot \frac{z}{a}.$$

Multiply these by $2 d_t x$, $2 d_t y$, $2 d_t z$ respectively, and add and integrate;

$$\therefore v^2 = 2 \int g dz + C = 2gz + C \dots\dots\dots (1).$$

Again, multiply them by x , y , z respectively and add;

$$\therefore x d_t^2 x + y d_t^2 y + z d_t^2 z = gz - \frac{Ra}{m}.$$

But the equation of the sphere being twice differentiated gives

$$x d_t^2 x + y d_t^2 y + z d_t^2 z + (d_t x)^2 + (d_t y)^2 + (d_t z)^2 = 0,$$

$$\text{or } x d_t^2 x + y d_t^2 y + z d_t^2 z = -v^2 = -2gz - C;$$

$$\therefore -2gz - C = gz - \frac{Ra}{m};$$

$$\therefore R = 3mg \cdot \frac{z}{a} + \frac{mC}{a} \dots\dots\dots (2).$$

Again, multiply the second equation of motion by x , the first by y and subtract and integrate;

$$\therefore x d_1 y - y d_1 x = h.$$

But by differentiating the equation to the surface once, we find

$$x d_1 x + y d_1 y = -x d_1 x.$$

Adding the squares of these two equations we obtain

$$(x^2 + y^2) \{ (d_1 x)^2 + (d_1 y)^2 \} = h^2 + x^2 (d_1 x)^2;$$

$$\therefore (a^2 - x^2) \{ v^2 - (d_1 x)^2 \} = h^2 + x^2 (d_1 x)^2;$$

$$\begin{aligned} \therefore a^2 (d_1 x)^2 &= (a^2 - x^2) v^2 - h^2 \\ &= (a^2 - x^2) (2gx + C) - h^2; \end{aligned}$$

$$\therefore t = \int_x \frac{a}{\sqrt{(a^2 - x^2) (2gx + C) - h^2}} \dots\dots (3).$$

Equation (1) gives the velocity, (2) gives the reaction of the surface which is equal and opposite to the pressure; and (3) gives the time, whenever the integration can be effected.

The position of the vertical plane DPC in which the body is situated, may be found from the equation

$$h = x d_1 y - y d_1 x = r^2 d_1 \theta = (a^2 - x^2) d_1 \theta;$$

$$\begin{aligned} \therefore \theta &= \int \frac{h}{a^2 - x^2} \\ &= \int \frac{h d_x t}{a^2 - x^2} \dots\dots\dots (4). \end{aligned}$$

In this equation we must substitute for $d_x t$ from (3).

169. The vertical velocity of the body = $d_1 x$, and whenever it reaches the highest and lowest points to which it rises and falls in the bowl, this = 0; hence the greatest and least depths will be two positive roots of the equation $d_1 x = 0$, or

$$0 = (a^2 - x^2) (2gx + C) - h^2;$$

$$\dots\dots \text{or } 0 = x^2 + \frac{C}{2g} x^2 - a^2 x + \frac{h^2 - C a^2}{2g}.$$

Now when the roots of an equation are all positive, its terms are alternately positive and negative. Consequently as the coefficient of x in this equation is necessarily negative, the roots are not all positive. But as the motion by hypothesis takes place in a bowl, where there must of necessity be a greatest and a least altitude; to which correspond *two* positive roots of this equation, all the roots are possible; and two of them are positive and one negative. Let therefore α be the greatest root, β the least, and γ the negative root;

$$\therefore a^3 (d_1 x)^2 = 2g (\alpha - x) (x - \beta) (x + \gamma).$$

Now for $\alpha - x$ write x' , x' being the altitude of the body above its *lowest* position;

$$\therefore a^3 (d_1 x')^2 = 2g x' (\alpha - \beta - x') (a + \gamma - x')$$

$$\frac{(2g)^{\frac{1}{2}}}{a} \cdot t = \int_{x'} \frac{-1}{\sqrt{(\alpha - \beta) x' - x'^2} \cdot \sqrt{a + \gamma - x'}}.$$

Now from the equation (B) in Art. 163 for the time of oscillation in a circular arc, we find that if t' be the time of descending from an altitude $\alpha - \beta$ to an altitude x' in a circular arc the diameter of which is $a + \gamma$, then

$$\frac{(2g)^{\frac{1}{2}}}{\frac{1}{2}(a + \gamma)} \cdot t' = \int_{x'} \frac{-1}{\sqrt{(\alpha - \beta) x' - x'^2} \cdot \sqrt{a + \gamma - x'}};$$

$$\therefore \frac{t}{a} = \frac{2t'}{a + \gamma},$$

$$\text{or } t = t' \cdot \frac{2a}{a + \gamma}.$$

If we consider the time of one oscillation to be the interval between leaving the highest position and returning to the highest position again, it is

$$= \pi \left\{ \frac{2a^2}{(a + \gamma)g} \right\}^{\frac{1}{2}} \cdot \left\{ 1 + \left(\frac{1}{2} \right)^2 \cdot \frac{a - \beta}{a + \gamma} + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \cdot \left(\frac{a - \beta}{a + \gamma} \right)^2 + \dots \right\} \dots \text{Art. 163.}$$

170. It will be most convenient to express the arbitrary constants in Art. 169, in terms of α , β , the greatest and least depths which the body attains in the course of its motion. This may be done as follows: since

$$(\alpha^2 - z^2)(2gz + C) - h^2 = 2g(\alpha - z)(z - \beta)(z + \gamma)$$

for all values of z , let $z = \alpha$;

$$\therefore (\alpha^2 - \alpha^2)(2g\alpha + C) = h^2;$$

$$\text{let } z = \beta, \quad \therefore (\alpha^2 - \beta^2)(2g\beta + C) = h^2;$$

$$\text{let } z = 0, \quad \therefore C\alpha^2 - h^2 = -2ga\beta\gamma.$$

From the first two we obtain,

$$C = -2g \cdot \frac{\alpha^3 + \alpha\beta + \beta^3 - \alpha^3}{\alpha + \beta},$$

$$h^2 = 2g \cdot \frac{(\alpha^2 - \alpha^2)(\alpha^2 - \beta^2)}{\alpha + \beta}.$$

From the third equation we find

$$\gamma = \frac{\alpha^2 + \alpha\beta}{\alpha + \beta};$$

$$\therefore \alpha + \gamma = \frac{\alpha^2 + 2\alpha\beta + \alpha^2}{\alpha + \beta}.$$

Hence the time of oscillation

$$\begin{aligned} &= \pi \left\{ \frac{2\alpha^2(\alpha + \beta)}{g(\alpha^2 + 2\alpha\beta + \alpha^2)} \right\}^{\frac{1}{2}} \cdot \left\{ 1 + \left(\frac{1}{2} \right)^2 \cdot \frac{\alpha^2 - \beta^2}{\alpha^2 + 2\alpha\beta + \alpha^2} \right. \\ &\quad \left. + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \cdot \left(\frac{\alpha^2 - \beta^2}{\alpha^2 + 2\alpha\beta + \alpha^2} \right)^2 + \dots \right\}. \end{aligned}$$

171. The body has both a vertical and a horizontal motion at the same time. Art. 169 shews us that its vertical motion bears a constant relation to that of a certain pendulum in a vertical plane; we must now examine its horizontal motion. The equation $h = xdy - ydx$ obtained in Art. 168, informs us that areas are described equably by OM about O . If therefore we suppose a body projected from

M and acted on by a proper force tending to O (Art. 122), this body will describe P 's horizontal orbit. Let F' be this required force, v' the velocity of M ;

$$\begin{aligned}\therefore v'^2 &= (d_t x)^2 + (d_t y)^2 \\ &= v^2 - (d_t z)^2 \\ &= \frac{x^2 (2gx + C) + h^2}{a^2}; \\ \therefore F' &= -v' d_t v' = -v' d_t v' \cdot d_t x \\ &= -\frac{3gx^2 + Cs}{a^2} \cdot \left(-\frac{r}{x}\right) \dots \because x^2 = a^2 - r^2 \\ &= \frac{3gr\sqrt{a^2 - r^2} + Cr}{a^2},\end{aligned}$$

It is not possible to find the equation to this orbit in exact terms; but if we suppose r to be exceedingly small in comparison of the radius of the bowl,

$$F' = \frac{3ga + C}{a^2} \cdot r \text{ nearly,}$$

which $\propto r$ nearly,

and therefore the orbit will be very nearly an ellipse, having its centre at O (Art. 124).

The apsidal distances may be found from the equation $d_t x = 0$, and are therefore equal to $\sqrt{a^2 - \alpha^2}$ and $\sqrt{a^2 - \beta^2}$. But the most singular and striking phenomenon attending the motion in a smooth bowl is the progression of the apsides, which the student may witness experimentally by suspending a small leaden bullet by a string. If it be made to perform eccentric conical revolutions, the motion of the apsides will be very apparent.

172. To find the value of the apsidal angle, when the oscillations are nearly circular, or a nearly equal to β .

We must find an approximate value of θ . From (4) we have

$$\begin{aligned}
 d_s \theta &= \frac{h}{a^2 - s^2} \cdot d_s t \\
 &= \frac{h}{a^2 - s^2} \cdot \frac{a}{\sqrt{(a^2 - s^2)(2gs + C) - h^2}}, \text{ from (3);} \\
 &= \frac{h}{a^2 - s^2} \cdot \frac{a}{\sqrt{2g(a-s)(s-\beta)(s+\gamma)}}, \\
 \therefore d_s \theta &= \frac{h}{a^2 - s^2} \cdot \frac{a}{\sqrt{2g(a+\gamma)(a-\beta-s')s'}}.
 \end{aligned}$$

In writing a for s in some terms, we have been guided by the rule that s' may be neglected in comparison of a , or γ ; but not in comparison with $a - \beta$ which is of the same order as s' . If now we put

$$n = \frac{(a^2 - \alpha^2) \sqrt{2g(a+\gamma)}}{ah},$$

we have

$$\begin{aligned}
 nd_s \theta &= \frac{1}{\sqrt{(a-\beta)s' - s'^2}}, \\
 \therefore n\theta &= \text{vers}^{-1} \frac{2s'}{a-\beta} = \text{vers}^{-1} \frac{2(a-s)}{a-\beta},
 \end{aligned}$$

no constant being added, because we suppose θ to be zero when $s = a$.

The particle moves from nearer to farther apse while s changes from a to β , and in that time therefore $n\theta$ increases from zero to π ,

$$\therefore \text{apsidal angle} = \frac{\pi}{n} = \frac{\pi}{2} \cdot \frac{2a}{(a^2 + 2a\beta + \alpha^2)^{\frac{1}{2}}} \cdot \left(\frac{a^2 - \beta^2}{a^2 - \alpha^2} \right)^{\frac{1}{2}}.$$

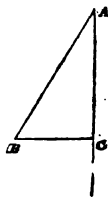
The excess of this angle above $\frac{\pi}{2}$ is the progression of the apse in half an oscillation. When α and β approach towards a , the apsidal angle approaches towards $\frac{\pi}{2}$ as its limit; but, when the oscillations are large, the progression of the apses is

very considerable: both of which results follow at once from the preceding value of the apsidal angle; from which also it is easy to shew that the apsides can never regress. The approximate equation of the orbit is

$$\frac{1}{r^2} = \frac{\left(\cos \frac{n\theta}{2}\right)^2}{a^2 - \alpha^2} + \frac{\left(\sin \frac{n\theta}{2}\right)^2}{a^2 - \beta^2}.$$

173. *Проб. A particle acted on by gravity, being suspended from a fixed point by a string, is made to describe a horizontal circle in such a manner that the string describes a conical surface. To find the time of revolution. This is called a conical pendulum.*

Let $AB = a$ be the string, m = the mass of the particle at B ; $BAC = \alpha$, AC being vertical; T = tension of the string, this would produce an accelerating force upon $m = \frac{T}{m}$ in the direction BA . Resolving this force perpendicular and parallel to the horizon, it appears that B is acted on by a force $g - \frac{T}{m} \cos \alpha$



in a direction perpendicular to the horizon, and $\frac{T}{m} \sin \alpha$ which always tends to the point C . By the question the former produces no motion,

$$\therefore 0 = g - \frac{T}{m} \cos \alpha \dots \dots \dots (1),$$

and the latter retains the body in a circle whose centre is C . But when a body moves in a circle about a force in its centre the centripetal force must be equal to the centrifugal force

$$= \frac{v^2}{BC} = \frac{v^2}{a \sin \alpha} \text{ in this case ;}$$

$$\begin{aligned} \therefore \frac{v^2}{a \sin \alpha} &= \frac{T}{m} \sin \alpha \\ &= g \cdot \frac{\sin \alpha}{\cos \alpha} \dots \text{ from (1) ;} \end{aligned}$$

$$\therefore v^2 = ag \cdot \frac{\sin^2 \alpha}{\cos \alpha}.$$

But since the motion in a circle is uniform,
the time of revolution

$$= \frac{\text{circumference}}{\text{velocity}} = \frac{2\pi a \sin \alpha}{v} = 2\pi \sqrt{\frac{a \cos \alpha}{g}}.$$

Since this result depends only on $a \cos \alpha$ or AC , it follows that whatever be the length of the string, providing the altitude of the cone described remain the same, the time will be constant.

It is evident that this problem is a particular case of the motion in a hemispherical bowl before discussed, viz. when $\alpha = \beta$.

CHAPTER VI.

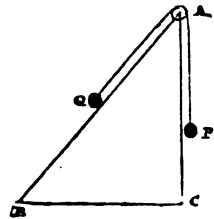
ON THE MOTION OF A SYSTEM OF PARTICLES.

174. DEF. BODIES constitute a dynamical system when they exercise an influence upon each other's motion, either directly, or indirectly through rods, strings, or any machinery by which they are connected.

175. In the preceding pages the motion of a single particle has been considered; the same principles apply to the motion of a system of particles, if we reckon among the impressed forces the tensions, mutual forces, reactions, &c. which are exerted upon each particle, as in the following example.

PROB. *Two weights P, Q are connected by a flexible string without weight passing over the top of a smooth inclined plane AB. P descends drawing up Q. It is required to determine the motion.*

Let P, Q represent the masses of the weights, a the length of the string, $x = AP$, $x' = AQ$; g = the force of gravity; α = the angle ABC , the inclination of the plane; T = the tension of the string. Then P is acted upon by the accelerating force g downwards, and $\frac{T}{P}$ (Art. 23) upwards; consequently for the motion of P we have the equation,



$$d_t^2 x = g - \frac{T}{P}.$$

Again, Q is acted upon by the force $g \sin \alpha$ down the plane, and $\frac{T}{Q}$ up the plane, and consequently for the motion of Q we have the equation,

$$d_t^2 x' = g \sin \alpha - \frac{T}{Q}.$$

To these we must join the geometrical relation $x + x' = a$, from which we obtain $d_t^2 x + d_t^2 x' = 0$ by differentiating twice;

$$\therefore g - \frac{T}{P} + g \sin \alpha - \frac{T}{Q} = 0, \text{ by substitution;}$$

$$\text{and } \therefore T = \frac{PQ}{P + Q} \cdot g (1 + \sin \alpha).$$

Hence the tension of the string is constant during the motion: its value, just found, being written in the preceding equations of motion, gives

$$d_t^2 x = \frac{P - Q \sin \alpha}{P + Q} \cdot g = -d_t^2 x'.$$

From which by integration the values of x and x' may be found in terms of t .

176. The preceding method may be generally applied with facility in simple cases of motion. It assumes that the force (T) exerted by the string upon P is the same as that which it exerts on Q . There is not much difficulty in admitting this assumption, it being a well known principle, that the tension of a string, when there is no friction, is the same throughout. But it is evident that in all cases when we have formed the differential equations of motion of all the particles of a system, they will contain the tensions, pressures, reactions, &c. proper to each and every particle; and as these are unknown forces it will be necessary to eliminate them from the equations of motion. But, as in the preceding question we eliminated the tension on the principle that the tension-force on each body is the same, so in every case there will be required some corresponding principle by which we may effect the elimination. Hence the necessity and utility of the following.

GENERAL PRINCIPLE. (1st Form.)

177. *The internal forces of a dynamical system are in equilibrium among themselves during the whole motion.*

Another form in which this is sometimes presented, and which is indeed frequently of more easy and more direct application, is the following.

GENERAL PRINCIPLE. (2nd Form.)

178. *The effective moving forces upon the molecules of a dynamical system, if their directions be reversed, will balance the external impressed forces.*

The principle exhibited in these two statements is commonly known as D'Alembert's Principle. Its truth may be rendered evident to the reader by the following considerations.

179. In a dynamical system neither the motion nor the change of motion of any one particle (speaking generally) is in the direction in which the resultant of the external forces are impressed upon it. (Ex. The body *Q* in Art. 175 moves up the plane, while the force of gravity is impressed upon it in a direction vertically downwards.) Each particle is under mechanical restraint, which prevents it from freely obeying the external forces. Now since these forces (which otherwise would wholly rule the motion of the particle) are partly resisted by the machinery and physical constitution of the system, we may resolve them into two sets; one set being that which is wholly resisted by the machinery, (including in this term, tensions, reactions ...) and the other being that which is wholly employed in producing the actual motion of each particle. (The former are called *forces lost*, the latter *effective forces*). The possibility of such a resolution of the impressed forces is evident. Now, it is manifest that the 'forces lost' have no influence on the motion of any particle, and if they were

removed the proper motion of the system would be continued by the action of the effective forces alone. Hence, if forces equal, and opposite, to the effective forces were applied to every particle, the system would cease to be accelerated or retarded; *i.e.* it would move as if acted on by no forces at all. Consequently, *the reversed effective forces exactly balance the impressed forces*; which is the principle as stated in 178.

Again, since the 'forces lost' have no share in producing the actual motion of the system, they are in equilibrium amongst themselves; for were they not in equilibrium, motion proper to their resultant would ensue. But *all* the motion is due to the effective forces: consequently, *the 'forces lost' (or the internal forces of the system) are in equilibrium amongst themselves during the whole motion*; which is the principle as stated in 177.

180. From the explanation just given, it will be seen that the effective forces and 'forces lost' are the *components* of the impressed forces; consequently, the 'forces lost' are the *resultants* of the impressed forces and *reversed* effective forces. (This will be evident from the parallelogram of forces; by consulting which the reader will see, that if a component be reversed the other component will then become the resultant of the reversed and original force). Hence, the impressed and reversed effective forces taken together are such as in every case will satisfy the statical conditions of equilibrium of the machine on which they act.

Hence, if $X_1, Y_1, Z_1, X_2, Y_2, Z_2, \dots$ be the *impressed* accelerating forces acting on the particles $m_1, m_2, \dots$ of a system, in the directions of three rectangular co-ordinate axes; the *effective* accelerating forces in the same directions are

$$d_t^3 x_1, \quad d_t^3 y_1, \quad d_t^3 z_1, \quad \text{upon } m_1$$

$$d_t^3 x_2, \quad d_t^3 y_2, \quad d_t^3 z_2, \quad \text{upon } m_2$$

.....

then the forces which balance each other on the machine are

$$\begin{aligned} & m_1(X_1 - d_t^2 x_1), \quad m_1(Y_1 - d_t^2 y_1), \quad m_1(Z_1 - d_t^2 z_1) \quad \text{upon } m_1 \\ & m_2(X_2 - d_t^2 x_2), \quad m_2(Y_2 - d_t^2 y_2), \quad m_2(Z_2 - d_t^2 z_2) \quad \text{upon } m_2 \\ & \dots\dots\dots \end{aligned}$$

181. For the purpose of reference it may be useful to put down here the six general equations of equilibrium of these forces.

$$\begin{aligned} 0 &= m_1(X_1 - d_t^2 x_1) + m_2(X_2 - d_t^2 x_2) + \dots = \Sigma \{m(X - d_t^2 x)\}, \\ 0 &= m_1(Y_1 - d_t^2 y_1) + m_2(Y_2 - d_t^2 y_2) + \dots = \Sigma \{m(Y - d_t^2 y)\}, \\ 0 &= m_1(Z_1 - d_t^2 z_1) + m_2(Z_2 - d_t^2 z_2) + \dots = \Sigma \{m(Z - d_t^2 z)\}; \\ 0 &= \{m_1(Z_1 - d_t^2 z_1)y_1 - m_1(Y_1 - d_t^2 y_1)z_1\} \\ &\quad + \{m_2(Z_2 - d_t^2 z_2)y_2 - m_2(Y_2 - d_t^2 y_2)z_2\} + \dots \\ &= \Sigma \{m(Zy - Yz) - m(yd_t^2 z - zd_t^2 y)\}, \\ \text{so,} \quad 0 &= \Sigma \{m(Xz - Zx) - m(zd_t^2 x - xd_t^2 z)\}, \\ \text{and} \quad 0 &= \Sigma \{m(Yx - Xy) - m(xd_t^2 y - yd_t^2 x)\}; \end{aligned}$$

or, as they may be more conveniently written

$$\Sigma (md_t^2 x) = \Sigma (mX) \dots\dots\dots (1),$$

$$\Sigma (md_t^2 y) = \Sigma (mY) \dots\dots\dots (2),$$

$$\Sigma (md_t^2 z) = \Sigma (mZ) \dots\dots\dots (3),$$

$$\Sigma \{m(yd_t^2 z - zd_t^2 y)\} = \Sigma \{m(Zy - Yz)\} \dots\dots (4),$$

$$\Sigma \{m(zd_t^2 x - xd_t^2 z)\} = \Sigma \{m(Xz - Zx)\} \dots\dots (5),$$

$$\Sigma \{m(xd_t^2 y - yd_t^2 x)\} = \Sigma \{m(Yx - Xy)\} \dots\dots (6).$$

182. D'Alembert's principle, embodied in these six equations, is sufficient for the solution of every problem in Dynamics that can be proposed. But, for facilitating their application, certain results have been derived from them, on

particular hypotheses ; which results, having a certain degree of generality, viz., the same degree of generality as the hypotheses on which they have been obtained, have received the appellation of *general principles*. These we shall now proceed to lay before the reader.

FIRST GENERAL PROBLEM.

183. *A system of free particles is subjected to no other forces than their mutual actions ; to determine the dynamical properties of every such system.*

Under the term *mutual action* we include attraction, repulsion and friction, and pressure by contact with each other, whether impulsive or not. In all cases which fall under this problem, for each moving force impressed upon a particle of the system there is an equal and opposite moving force (see Arts. 24, 26) impressed on some other particle : and, consequently, the sum of the components, of the impressed forces, in any direction is equal to zero : also the sum of their moments about any axis whatever will be equal to zero. Now (by Art. 178) the *effective* forces balance these forces, and therefore the effective forces are in equilibrium ; and the sums of their components in three directions must also be separately equal to zero ; and the sums of their moments about three fixed lines must also be separately equal to zero. Resolving them along the co-ordinate axes, and taking the moments about the same lines, (as in Art. 181) we have the six following equations,

$$\Sigma (m d_t^2 x) = 0, \quad \Sigma (m d_t^2 y) = 0, \quad \Sigma (m d_t^2 z) = 0 \dots\dots (A),$$

$$\Sigma \{m (y d_t^2 x - x d_t^2 y)\} = 0, \quad \Sigma \{m (x d_t^2 z - z d_t^2 x)\} = 0,$$

$$\Sigma \{m (x d_t^2 y - y d_t^2 x)\} = 0 \dots (B).$$

From these six equations we shall deduce three general principles.

- I. The conservation of the motion of the centre of gravity.
- II. The conservation of areas.
- III. The existence of an invariable plane.

1. THE CONSERVATION OF THE MOTION OF THE CENTRE OF GRAVITY.

184. *The centre of gravity of a system of particles subject only to mutual influence, if not at rest, moves uniformly through space in a straight line.*

We deduce this principle from the equations of Art. 183 marked (A). Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity of the system; then

$$\bar{x} \cdot \Sigma m = \Sigma (mx),$$

$$\therefore d_t^2 \bar{x} \cdot \Sigma m = \Sigma (m d_t^2 x) \text{ by differentiation,}$$

$$= 0, \text{ by equations (A),}$$

$$\therefore d_t^2 \bar{x} = 0.$$

$$\text{Similarly } d_t^2 \bar{y} = 0, \text{ and } d_t^2 \bar{z} = 0.$$

By integrating these equations we find

$$d_t \bar{x} = A, \quad d_t \bar{y} = B, \quad d_t \bar{z} = C \dots\dots (1);$$

$$\text{and } \bar{x} = At + a, \quad \bar{y} = Bt + b, \quad \bar{z} = Ct + c;$$

or, if we eliminate t from the last set of equations,

$$\frac{\bar{x} - a}{A} = \frac{\bar{y} - b}{B} = \frac{\bar{z} - c}{C} \dots\dots (2).$$

Equations (1) shew that the centre of gravity of the system is either at rest, or moves uniformly in space; and equations (2) shew that if the centre of gravity be not at rest its path in space is a straight line.

2. THE CONSERVATION OF AREAS.

185. *In a system of particles, subject only to mutual influence, the sum of the products of the mass of each into the projection, on any fixed plane, of the area swept out by its radius vector drawn to that point is proportional to the time.*

We deduce this principle from the equations of Art. 183, marked (B). By integrating the first of them we have

$$\Sigma \{m(y d_t x - x d_t y)\} = \text{a constant, } h' \text{ suppose.}$$

Now if A_{x_1} be the polar area described on the plane yx by the projection of the radius vector joining m_1 and the origin, we know from the Differential Calculus that

$$2d_1 A_{x_1} = y_1 d_1 x_1 - x_1 d_1 y_1;$$

$$\text{and } \therefore 2 \Sigma(m d_1 A_x) = h'.$$

And this being integrated gives

$$\Sigma(m A_x) = \frac{1}{2} h' t.$$

Similarly from the other two equations of the set (B) we have

$$\Sigma(m A_y) = \frac{1}{2} h'' t, \quad \Sigma(m A_z) = \frac{1}{2} h''' t;$$

no constants being added after integration, because the areas are all supposed to commence when $t = 0$.

Now the positions of the co-ordinate planes being perfectly arbitrary, any one of them may be assumed to coincide with a given fixed plane; and consequently these equations establish the property above enunciated.

3. THE INVARIABLE PLANE.

186. *In a system of particles, subject only to mutual influence, there is a certain plane, the position of which in space (as calculated from the motions of the particles referred to any arbitrary system of co-ordinate axes) is invariable.*

We have seen that in such a system of particles

$$\Sigma(m A_x) = \frac{1}{2} h' t, \quad \Sigma(m A_y) = \frac{1}{2} h'' t, \quad \Sigma(m A_z) = \frac{1}{2} h''' t$$

represent

$\{\Sigma(m A_x)\}^2 + \{\Sigma(m A_y)\}^2 + \{\Sigma(m A_z)\}^2$ by $\{\Sigma(m A)\}^2$,
and $h'^2 + h''^2 + h'''^2$ by h^2 ; and through the fixed point draw a plane, the inclinations of which to the co-ordinate planes yz , xz , xy are λ , μ , ν ; which are such that

$$\cos \lambda = \frac{\Sigma(m A_x)}{\Sigma(m A)}, \quad \cos \mu = \frac{\Sigma(m A_y)}{\Sigma(m A)}, \quad \cos \nu = \frac{\Sigma(m A_z)}{\Sigma(m A)}.$$

This plane is invariable in position, for by substitution we find $\Sigma(mA) = \frac{1}{2}ht$, and therefore

$$\cos \lambda = \frac{h'}{h}, \quad \cos \mu = \frac{h''}{h}, \quad \cos \nu = \frac{h'''}{h},$$

which being independent of t are invariable, and therefore the plane above drawn is fixed in position during the motion of the system.

187. In Art. 185 it was proved that

$$h' = \Sigma \{m(yd_1x - xd_1y)\}.$$

Now if s'_1 be the projection of m_1 's path on the plane yx , and p'_1 be the perpendicular dropt from the origin upon that projected path, we know from the Differential Calculus that

$$h' = \Sigma \{m(yd_1x - xd_1y)\} = \Sigma (mp'd_1s').$$

Now $d_1s'_1$ is the velocity of m , parallel to plane yx ; hence $m_1d_1s'_1$ is the momentum of m_1 , and $m_1d_1s'_1p'_1$ is the moment of m_1 's momentum about the axis of x : consequently h' = the moment of the momentum of the whole system about the axis of x , which is therefore invariable. The same property may be proved with respect to the other axes of co-ordinates; and as they are arbitrary in position, the moment of the momentum of the whole system, about any axis whatever, passing through a fixed point, is invariable during the motion.

188. *The axis, about which the moment of the momentum of the whole system is the greatest possible, is at right angles to the invariable plane.*

For h' , h'' , h''' , being the sums of the projections of certain areas multiplied into certain constants, the sum of the projections of the same on any plane will be equal to the sum of the projections of h' , h'' , h''' , on that plane. Hence, if a line through the origin make angles θ , ϕ , ψ with the co-ordinate axes, the moment of the momenta about this line, which is equal to the sum of the projections of h' , h'' , h''' on a plane at right angles to that line, is

$$h' \cos \theta + h'' \cos \phi + h''' \cos \psi.$$

This is, then, the expression for the moment of the momentum about that line; and it is to be a maximum by the variation of θ , ϕ , ψ , subject to the condition

$$\cos^2 \theta + \cos^2 \phi + \cos^2 \psi = 1.$$

Hence we find, by the usual process for finding maxima and minima values of a function of two independent variables,

$$\cos \theta = \frac{h'}{h}, \quad \cos \phi = \frac{h''}{h}, \quad \cos \psi = \frac{h'''}{h}.$$

Hence, $\theta = \lambda$, $\phi = \mu$, $\psi = \nu$; and the line makes the same angles with the co-ordinate axes, as the invariable plane with the co-ordinate planes. Consequently they are at right angles to each other.

REMARKS ON THE FOREGOING RESULTS.

189. I. Since the properties of the motion of the centre of gravity (Art. 184) have the same degree of generality as the equations (A) from which they are derived, they will hold in all systems, (even though not included in the enunciation of Art. 183) to which those equations apply, and therefore in the following cases:

1. If any of the particles, whether elastic or not, impinge against each other. For, the pressures exerted during impact are equal and opposite.

2. If any particles be connected by strings, elastic or inelastic, whether kept always stretched or not. For, the actions of each string on the particles which it connects will be equal and opposite.

3. If the particles be connected in any way, providing there be not a fixed point in any part of the connecting machinery. For, the pressures exerted on the particles by the connecting apparatus are in equilibrium (Art. 177).

4. If the particles be under the influence of any external system of forces, which either balance each other, or are reducible to a resultant couple. For, such a system of forces have

no components parallel to the co-ordinate axes, and therefore do not affect equations (*A*).

190. II. It will have been noticed in the demonstration of Art. 184, that the velocity of the centre of gravity parallel to one of the co-ordinate axes is not dependent upon the equations of motion with reference to the other axes. Hence, if one of those equations be true, the corresponding property of the motion of the centre of gravity is true. Now the directions of the co-ordinate axes are quite arbitrary. Hence, if the system be under the action of external forces, and a line can be drawn in space such that the sum of the components of the impressed forces in that direction be always equal to zero, the centre of gravity will either have no motion in that direction, or its motion will be uniform. Ex. gr. If a system be subject only to the force of gravity and vertical re-actions, the horizontal motion of the centre of gravity will be uniform.

191. III. The principle of the conservation of areas, and the existence of an invariable plane, depend only on the truth of the equations (*B*). They will therefore be true in all systems in which those equations hold.

1, 2. Hence, cases 1, 2, of Art. 189, will hold here also, since in both of them the sum of the momenta about any axis vanishes. Also

3. If the particles be connected in any way, even if there be a fixed point, providing that point be the pole about which the areas are described. For, the sum of the momenta about any line passing through the pole vanishes.

4. If the particles be under the influence of any external system of forces, which either balance each other, or have a single resultant force which passes through the pole of areas. For such a system having no resultant couple, the sum of the momenta about any one of the lines through the pole must be equal to zero.

192. IV. It will be noticed in the demonstration of Art. 185, that each of the three equations $\Sigma (m A_i) = \frac{1}{2} h' t$,

$\Sigma(m A_y) = \frac{1}{2} h'' t$, $\Sigma(m A_z) = \frac{1}{2} h''' t$, is obtained independently of the other two. The conservation of areas, as we have enunciated the principle, requires that these should be all true together: but any one of them may be true without necessitating the truth of the other two. The property expressed by one of these equations will be true of all systems of particles in which the corresponding equation of the set (B) is true. Now the positions of the origin and co-ordinate axes are arbitrary. Hence if there be in space a line about which the impressed forces balance each other, the principle of conservation of areas is true for any plane drawn perpendicular to that line, the pole of areas being the point where the plane and line intersect.

193. V. *The principle of the conservation of areas, and the existence of the invariable plane, are properties which hold good when the centre of gravity of the system is taken as the pole of areas, even though the centre of gravity be not a stationary point.*

For, let x_1', y_1', z_1' be the co-ordinates of any particle m_1 , referred to the centre of gravity of the system as origin and axes parallel to the fixed axes in space. Then $x_1 = \bar{x} + x_1'$, &c., and substituting in the first of equations (B) we have

$$0 = \Sigma [m \{ (\bar{y} + y') d_t^2 (\bar{z} + z') - (\bar{z} + z') d_t^2 (\bar{y} + y') \}].$$

Now observing that $\bar{y}, \bar{z}$ are not subject to the symbol Σ , and that from the last of equations A, and because $\Sigma(m z') = 0$,

$$0 = \Sigma(m d_t^2 z) = (\Sigma m) \cdot d_t^2 \bar{z}; \text{ and } \Sigma(m d_t^2 z') = 0;$$

and similarly

$$0 = \Sigma(m d_t^2 y) = (\Sigma m) \cdot d_t^2 \bar{y}; \text{ and } \Sigma(m d_t^2 y') = 0,$$

this becomes

$$0 = \Sigma \{ m (y' d_t^2 z' - z' d_t^2 y') \} + \bar{y} \Sigma (m d_t^2 z') - \bar{z} \Sigma (m d_t^2 y').$$

Hence

$$0 = \Sigma \{ m (y' d_t^2 z' - z' d_t^2 y') \}.$$

Hence the form of the first of the equations (B) is not changed when the centre of gravity is made the origin of

co-ordinates; and as the same may be proved of the other two equations, the properties which have been proved in Arts. 185, 186 are true when the centre of gravity is the pole of areas, even though it be in motion.

194. *If the particles of a system be acted on by no other forces than their mutual attractions, the expression $\Sigma \{m(Xdx + Ydy + Zdz)\}$ will be an exact differential.*

For let P' be the attractive pressure exerted between m_1, m_2 , the distance of which $= r'$. Then m_1 is urged towards m_2 with an accelerating force $= \frac{P'}{m_1}$; which produces an accelerating force in the direction of $x = \frac{P'}{m_1} \cdot \frac{x_2 - x_1}{r'} = - \frac{P'}{m_1} \cdot d_{x_1} r'$;

$$\therefore r'^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Similarly, if P'' be the attractive pressure between m_1 and m_3 the distance of which $= r''$, it produces an accelerating force upon m_1 in the direction of $x = - \frac{P''}{m_1} \cdot d_{x_1} r''$, and so on;

$$\therefore X_1 = - \frac{P'}{m_1} d_{x_1} r' - \frac{P''}{m_1} d_{x_1} r'' - \frac{P'''}{m_1} d_{x_1} r''' - \dots$$

$$\therefore m_1 X_1 dx_1 = - (P' d_{x_1} r' dx_1 + P'' d_{x_1} r'' dx_1 + P''' d_{x_1} r''' dx_1 + \dots)$$

Similar expressions may be obtained for $m_1 Y_1 dy_1, m_1 Z_1 dz_1$. Adding the results, we find

$$\begin{aligned} m_1 (X_1 dx_1 + Y_1 dy_1 + Z_1 dz_1) &= - P' (d_{x_1} r' dx_1 + d_{y_1} r' dy_1 + d_{z_1} r' dz_1) \\ &\quad - P'' (d_{x_1} r'' dx_1 + d_{y_1} r'' dy_1 + d_{z_1} r'' dz_1) \\ &\quad - \&c. \end{aligned}$$

Now in the expression for $m_2 (X_2 dx_2 + Y_2 dy_2 + Z_2 dz_2)$ similarly obtained, there will be a term $- P' (d_{x_2} r' dx_2 + d_{y_2} r' dy_2 + d_{z_2} r' dz_2)$ which being added to the first term in the above expression for $m_1 (X_1 dx_1 + Y_1 dy_1 + Z_1 dz_1)$, gives $- P' (d_{x_1} r' dx_1 + d_{y_1} r' dy_1 + d_{z_1} r' dz_1 + d_{x_2} r' dx_2 + d_{y_2} r' dy_2 + d_{z_2} r' dz_2)$ which is equal to $- P' dr'$; when all the expressions for $m_3 (X_3 dx_3 + \dots)$, $m_4 (X_4 dx_4 + \dots)$ &c. are formed, and added to those for

$m_1 (X_1 dx_1 + \dots)$, $m_2 (X_2 dx_2 + \dots)$, every term may be added to another, by which the sum may be simplified in the same manner as that involving P' ;

$$\therefore \Sigma \{m (X dx + Y dy + Z dz)\} = -P' dr' - P'' dr'' - P''' dr''' - \dots$$

Now as P' is a function of r' alone, $P' dr'$ is an exact differential; the same may be said of every term in the right-hand member. Hence the left-hand member, being reducible to it, is also an exact differential.

SECOND GENERAL PROBLEM.

195. *A free system of rigidly connected particles (or, a rigid body) is under the influence of any given external forces; to determine the dynamical properties of every such system.*

Let X_1, Y_1, Z_1 be the external accelerating forces impressed on m_1 , a similar notation being employed for the forces acting on the other particles. Then the equations proper for such a system as we are now considering, are the six equations of Art. 181, viz.

$$\begin{aligned} \Sigma (m d_t^2 x) &= \Sigma (m X), & \Sigma (m d_t^2 y) &= \Sigma (m Y), \\ & & \Sigma (m d_t^2 z) &= \Sigma (m Z) \dots (C), \\ \left. \begin{aligned} \Sigma \{m (y d_t^2 z - z d_t^2 y)\} &= \Sigma \{m (Z y - Y z)\} \\ \Sigma \{m (z d_t^2 x - x d_t^2 z)\} &= \Sigma \{m (X z - Z x)\} \\ \Sigma \{m (x d_t^2 y - y d_t^2 x)\} &= \Sigma \{m (Y x - X y)\} \end{aligned} \right\} \dots (D). \end{aligned}$$

From these six equations we shall be able to prove the independence of the motions of translation and rotation of a rigid body acted on by any forces.

I. THE CONSERVATION OF THE MOTION OF TRANSLATION.

196. *The motion of the centre of gravity of a free rigid body, is the same as if all the external forces were transferred to that point without changing their directions.*

This property is proved from the three equations marked (C) as follows. Let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity of the body. Then

$$\bar{x} \Sigma m = \Sigma (m x);$$

$$\therefore \Sigma m \cdot d_t^2 \bar{x} = \Sigma (m d_t^2 x) = \Sigma (m X).$$

$$\text{So } \Sigma m \cdot d_t^2 \bar{y} = \Sigma (m d_t^2 y) = \Sigma (m Y),$$

$$\text{and } \Sigma m \cdot d_t^2 \bar{z} = \Sigma (m d_t^2 z) = \Sigma (m Z).$$

Now these are the equations of motion of the mass Σm , acted on by the forces $\Sigma (m X)$, $\Sigma (m Y)$, $\Sigma (m Z)$ applied at its centre of gravity: consequently, the motion of the centre of gravity of the body is the same as if the whole mass were collected there, and all the forces applied directly at that point without changing their directions.

II. THE CONSERVATION OF MOTION OF ROTATION.

197. *The motion of rotation of a free rigid body is the same as if the centre of gravity were a fixed point.*

This property is proved as follows.

Transpose the origin of co-ordinates as in Art. 193, to the centre of gravity by writing $\bar{x}+x'$, $\bar{y}+y'$, $\bar{z}+z'$ for x , y , z . Then $\Sigma \{m (y d_t^2 x - x d_t^2 y)\}$ becomes

$$\begin{aligned} & \Sigma \{m (\bar{y} + y') d_t^2 (\bar{z} + z') - m (\bar{z} + z') d_t^2 (\bar{y} + y')\} \\ &= (\Sigma m) (\bar{y} d_t^2 \bar{z} - \bar{z} d_t^2 \bar{y}) + d_t^2 \bar{z} \Sigma (m y') - d_t^2 \bar{y} \Sigma (m z') + \bar{y} \Sigma (m d_t^2 z') \\ & \quad - \bar{x} \Sigma (m d_t^2 y') + \Sigma \{m (y' d_t^2 z' - z' d_t^2 y')\}. \end{aligned}$$

Now because the centre of gravity is the origin of co-ordinates,

$$\Sigma (m y') = 0, \quad \text{and} \quad \Sigma (m z') = 0;$$

$$\therefore \Sigma (m d_t^2 y') = 0, \quad \text{and} \quad \Sigma (m d_t^2 z') = 0.$$

Also by the last article,

$$(\Sigma m) (\bar{y} d_t^2 \bar{z} - \bar{z} d_t^2 \bar{y}) = \bar{y} \Sigma (m Z) - \bar{z} \Sigma (m Y).$$

These being substituted in the above expression we find

$$\begin{aligned} \Sigma \{m (y d_t^2 x - x d_t^2 y)\} &= \bar{y} \Sigma (m Z) - \bar{z} \Sigma (m Y) \\ & \quad + \Sigma \{m (y' d_t^2 z' - z' d_t^2 y')\}. \end{aligned}$$

Also, $\Sigma \{m(Zy - Yz)\}$ the right-hand member of the first of equations (D) becomes

$$\begin{aligned} & \Sigma \{mZ(\bar{y} + y') - mY(\bar{z} + z')\} \\ &= \bar{y}\Sigma(mZ) - \bar{z}\Sigma(mY) + \Sigma\{m(Zy' - Yz')\}. \end{aligned}$$

Equating this with the right side of the last equation, we find

$$\Sigma \{m(y'd_i^2 z' - z'd_i^2 y')\} = \Sigma \{m(Zy' - Yz')\}.$$

Now the equation is of precisely the same *form* as the corresponding general equation in (D); but we observe that it does not, either directly or indirectly, involve the position (or co-ordinates) of the centre of gravity, or any forces acting on that point. Hence it represents a motion which is in no way dependent on the state of rest or motion of the centre of gravity, or on any forces which act upon it. The same may be shewn respecting the remaining equations of (D). Consequently, whether the centre of gravity be free or forcibly restrained in any manner, the motion represented by equations (D) (that is, the rotatory motion) will be the same.

198. We see then that of the six equations which express the entire motion of a rigid system, the three marked (C) represent only a motion of translation, (a motion which is common to every particle, being equal and parallel to that of the centre of gravity;) and the three marked (D) represent only angular motion about the centre of gravity.

199. We know from *Statics* that the equations (D) arise from the action of couples only, or forces which are equivalent to couples. Now since a couple consists of two equal forces acting in opposite directions, if the forces which furnish equations (D) were all applied to the centre of gravity they would two and two destroy each other, and would therefore not appear in the equations of (196) for the motion of the centre of gravity. And in a similar way the forces in (196), acting at the centre of gravity, cannot appear in the equations of (197). Hence the motions which are represented by (C) are quite distinct from those represented by (D). Either may exist or be changed in any manner without affecting the other. In other words,

The motions of translation and rotation are independent.

From what has been stated it appears that a couple applied to a body cannot move the centre of gravity.

THIRD GENERAL PROBLEM.

200. *A mixed system of bodies, either free or connected in any manner, and constrained by smooth fixed curve lines, surfaces, rigid rods, inextensible strings, &c.; to determine equations for the motion of every such system free from tensions and reactions.*

We are enabled to do this by the principle of virtual velocities. For, when the effective forces have been reversed, the forces then acting on the system including tensions of rods and cords, and reactions of curve lines and surfaces, are such as would maintain equilibrium. Now the peculiarity of the principle of virtual velocities is that it furnishes an equation free from tensions, &c., and, consequently, if applied to a dynamical system, will furnish an equation containing only impressed and effective forces, without tensions and reactions. Let $\delta x_1, \delta y_1, \delta z_1$, be any small *arbitrary* displacements which may be communicated to the particle m_1 , without deranging the system, *i. e.* without destroying contacts of particles with curve lines, &c.; $\delta x_2, \delta y_2, \delta z_2, \delta x_3, \delta y_3, \delta z_3 \dots$ similar quantities for $m_2, m_3, \dots$ Then by the principle of virtual velocities

$$\Sigma \{ m (X - d_t^2 x) \delta x + m (Y - d_t^2 y) \delta y + m (Z - d_t^2 z) \delta z \} = 0.$$

201. The great advantage of this equation consists in this, that it does not contain any terms depending on reactions and tensions. If the parts of the system be so connected as to render it impossible to displace one particle without disturbing *all* the others, it is obvious the virtual velocities for m_1 being assumed, those of all the other particles will be determinable by geometrical relations, and when so determined being written in the above equation, the coefficients of the arbitrary displacements being separately equated to zero will furnish the equations of motion of the system. It will, however, sometimes be possible to disturb one part of the system without

effecting other parts. In this case the above equation will furnish as many sets of equations of motion as there are independent parts of the system. Wherefore in using the above equation we must find, from the geometrical properties of the system, as many of the quantities $\delta x_1, \delta x_2, \delta x_3, \dots \delta y_1, \delta y_2, \delta y_3, \dots \delta z_1, \delta z_2, \delta z_3, \dots$ in terms of the others as possible, and substitute them in the equation, and then equate to zero the coefficients of each arbitrary displacement that remains. The resulting equations will be all the equations of motion of the system.

From the preceding equation we can deduce a principle of very extensive application, called

THE CONSERVATION OF VIS VIVA.

202. DEF. The *Vis Viva* of a particle is the product of its mass into the square of its velocity. The principle of the conservation of vis viva is the following :

203. *If a system of particles which are either connected or unconnected, and either constrained to move on given curve lines or not, be in motion by the action of external forces, which are not impulsive; then the change of vis viva of the whole system, in passing from one given position to another, is independent of the form of the paths taken by the particles; and it is the same as if the connection of the system had been dissolved, and each particle had been suffered to move freely from its former to its latter position.*

Using the same notation as before, we have

$$\Sigma \{m(X - d_t^2 x)\delta x + m(Y - d_t^2 y)\delta y + m(Z - d_t^2 z)\delta z\} = 0;$$

in which equation $\delta x_1, \delta y_1, \delta z_1, \dots$ represent *any possible* displacements which are consistent with the arrangement of the system. Now the displacements which the particles actually receive in a very small time τ , by reason of the actual motion of the system, are evidently among the possible displacements, and as a particular case may be written for $\delta x_1, \delta y_1, \delta z_1 \dots$ in

the above equation. But by Taylor's Theorem, taking in only the first order of small terms, (to which order only the principle of virtual velocities extends,) these are

$$d_1 x_1 \cdot \tau, \quad d_1 y_1 \cdot \tau, \quad d_1 z_1 \cdot \tau \quad \text{for } m_1$$

$$d_1 x_2 \cdot \tau, \quad d_1 y_2 \cdot \tau, \quad d_1 z_2 \cdot \tau \quad \text{for } m_2$$

.....

Making these substitutions and dividing by τ we obtain

$$0 = \Sigma \{m(X - d_1^2 x) d_1 x + m(Y - d_1^2 y) d_1 y + m(Z - d_1^2 z) d_1 z\},$$

or, by transposition,

$$\begin{aligned} \Sigma \{m(d_1 x d_1^2 x + d_1 y d_1^2 y + d_1 z d_1^2 z)\} \\ = \Sigma \{m(X d_1 x + Y d_1 y + Z d_1 z)\}. \end{aligned}$$

If now $v_1, v_2, v_3, \dots$ be the velocities of $m_1, m_2, m_3, \dots$ respectively,

$$v_1^2 = (d_1 x_1)^2 + (d_1 y_1)^2 + (d_1 z_1)^2,$$

$$\text{and } \therefore v_1 d_1 v_1 = d_1 x_1 d_1^2 x_1 + d_1 y_1 d_1^2 y_1 + d_1 z_1 d_1^2 z_1.$$

Hence $\Sigma(mv dv) = \Sigma \{m(X d_1 x + Y d_1 y + Z d_1 z)\}$; and this being multiplied by 2 and integrated gives

$$\Sigma(mv^2) = 2 \Sigma \{m \int (X dx + Y dy + Z dz)\} + C \dots (A).$$

This is called *the equation of vis viva*: and it is to be remarked that it contains no terms depending on the connection of the parts of the system, or on the reactions of the curve-lines and surfaces along which the particles are subjected to move. In Art. 144 we have proved that $Xdx + Ydy + Zdz$ is integrable *per se* for all forces in nature; hence the form of the integral of the right side of equation (A) and its value between given limiting positions will not depend upon the forms of the paths which the particles are constrained to take. Let $f(x, y, z) = \int (Xdx + Ydy + Zdz)$, and suppose $abcV$ the values of $xyzv$ in the first position of the system, and $a'b'c'V'$ corresponding quantities at the second position; then

$$\Sigma(mV'^2) - \Sigma(mV^2) = 2 \Sigma \{mf(a', b', c') - mf(a, b, c)\},$$

which expressed in words is the property enunciated at the head of this article.

204. If $a'b'c'$ be equal to abc respectively, then

$$\Sigma(m V'^2) = \Sigma(m V^2) :$$

that is, the vis viva of a system is always the same whenever it occupies a given position.

205. The equation (A) is of such extensive application in dynamical questions, that it may be proper to make some further remarks upon it. It differs from the equation of (200), for in that the displacements $\delta x, \delta y, \delta z$, &c. are arbitrary, at least so far as the condition of the system will allow : but in this dx, dy, dz , &c. being the actual spaces described, are not arbitrary. We cannot therefore from (A) derive any other equation of motion ; while, as we have shewn, the equation of (200) will furnish all the necessary equations of motion of the system.

206. In forming the right-hand member of equation (A) we need not reckon among the forces $X_1 Y_1 Z_1, X_2 Y_2 Z_2, \dots$ those which arise from the mutual action of the parts of the system ; such as tension, and the reaction of curves on which any of the particles move ; nor, in a word, any force which would disappear in forming the equation of virtual velocities. Hence any force may be neglected, of whatever nature it be, which acts on a point of the system in a direction at right angles to the motion of that point ; for the virtual velocity of such a point is zero, and in forming the equation of virtual velocities the corresponding term would on that account vanish from the expression.

Of this nature is *friction*, when it causes a body to *roll without sliding*. Friction however may *not* be neglected when it opposes a *sliding* motion, inasmuch as in that case the virtual velocity of the point on which it acts is not zero. For a similar reason if the system move in a resisting medium, the resistance must be reckoned amongst the impressed forces.

207. It is to be remarked also that a mutual pressure vanishes only when it takes place between two particles whose

virtual velocities are equal and have contrary algebraic signs. This will always be the case with the tensions of *inextensible rods, and strings*. But the tension of an *elastic* string, or of a *flexible* rod; and the pressures which measure the mutual attraction of two bodies will not necessarily vanish, because though they are equal (Art. 29), yet the virtual velocities of the bodies would not necessarily be equal, nor necessarily have opposite signs.

208. From this we shall understand why the equation of vis viva is not always true in the case of bodies impinging against each other. For though the pressure is mutual and the virtual velocities of the bodies have generally opposite signs, yet these velocities are equal only at the moment of greatest compression, (Art. 26) after which they become unequal again until the bodies finally separate. Art. 42, Case 2, shews that vis viva is lost by collision, unless the bodies are perfectly elastic. By putting $\lambda = 0$ in the equation just quoted, we perceive that vis viva is lost while the bodies are being compressed; and since λ enters that part of the equation which represents the total loss with a negative sign it is clear that a portion of the lost vis viva is regained during the subsequent expansion. Hence we may state as a general principle that *vis viva is diminished by collision, and increased by explosion*. When the elasticity is perfect the loss and gain are equal.

209. If the system pass through a position of equilibrium, then by the principle of *virtual velocities*,

$$\Sigma \{m(Xdx + Ydy + Zdz)\} = 0,$$

$$\text{and } \therefore d\Sigma(mv^2) = 0,$$

that is, $\Sigma(mv^2)$ is a maximum or minimum. To ascertain whether it be a maximum or a minimum, we may reason thus.

If the position be one of *stable* equilibrium, the forces tend to prevent the system from passing out of that position; that is, they have a tendency to reduce the system to a state of rest, and consequently to *diminish* the vis viva, which therefore has its maximum value in that position.

If the position be one of *unstable* equilibrium the tendency is to carry the body out of it, and to *increase* the vis viva, which therefore has a minimum value in that position.

If the position be one of *neuter* equilibrium there is no tendency either to increase or diminish the vis viva, which therefore remains constant.

Hence *in passing through a position of equilibrium the vis viva is a maximum, minimum, or neither; according as the position is one of stable, unstable, or neuter equilibrium.*

210. *If the system be acted on only by the accelerating force of gravity, (or any constant force) the right hand member of the equation of vis viva is equal to twice the force into the whole mass of the system multiplied by the space described, from the position of rest, by the common centre of gravity, in the direction of the force.*

For let g be the force, and suppose it to act parallel to the axis of x ; then

$$\begin{aligned} X &= g, \quad Y = 0, \quad \text{and} \quad Z = 0; \\ \therefore 2 \sum \{m\} (X dx + Y dy + Z dz) &= 2 \sum \{mg dx\} \\ &= 2g \sum (mx) + \text{constant} \\ &= 2g (\sum m) \cdot \bar{x} + \text{constant} \\ &= 2g (\sum m) \cdot (\bar{x} - \bar{a}), \end{aligned}$$

$\bar{a}$ being the value of $\bar{x}$ at the beginning of the motion, and consequently $\bar{x} - \bar{a}$ is the whole space described from rest by the centre of gravity in a direction parallel to the force.

In calculating the vis viva of a body the following property will often be found of great use.

211. *If a body move in any manner whatever, its vis viva at any instant is equal to the vis viva of the whole mass supposed collected at its centre of gravity + the vis viva round a parallel axis through the centre of gravity; or, in other words, it is equal to*

The vis viva due to translation + the vis viva due to rotation.

Since the vis viva does not depend upon the position of the co-ordinate axes, we are at liberty to take the axis of x parallel to the axis of rotation.

Let r_1 be the distance of any particle m_1 from a line passing through the centre of gravity parallel to the axis of x ; and let θ_1 be the inclination of r_1 to the plane xs ; then

$$x_1 = \bar{x} + r_1 \cos \theta_1, \quad \therefore d_t x_1 = d_t \bar{x} - r_1 \sin \theta_1 \cdot d_t \theta_1,$$

$$y_1 = \bar{y} + r_1 \sin \theta_1, \quad \therefore d_t y_1 = d_t \bar{y} + r_1 \cos \theta_1 \cdot d_t \theta_1,$$

$$z_1 = \bar{z} + \text{constant}, \quad \therefore d_t z_1 = d_t \bar{z};$$

$$\begin{aligned} \therefore v_1^2 &= (d_t x_1)^2 + (d_t y_1)^2 + (d_t z_1)^2 \\ &= (d_t \bar{x})^2 + (d_t \bar{y})^2 + (d_t \bar{z})^2 - 2 d_t \bar{x} \cdot d_t \theta_1 \cdot r_1 \sin \theta_1 \\ &\quad + 2 d_t \bar{y} \cdot d_t \theta_1 \cdot r_1 \cos \theta_1 + r_1^2 (d_t \theta_1)^2. \end{aligned}$$

Similar equations may be proved for $v_2^2, v_3^2, v_4^2, \dots$ multiplying them respectively by $m_1, m_2, m_3, m_4, \dots$ and adding, observing that $d_t \bar{x}, d_t \bar{y}, d_t \bar{z}$, are the same for all; and that $d_t \theta_1 = d_t \theta_2 = d_t \theta_3 = \dots$ because the body is rigid; and that $\Sigma(mr \sin \theta) = 0, \Sigma(mr \cos \theta) = 0$, by the nature of the centre of gravity, we obtain

$$\Sigma(mv^2) = \Sigma[m \{ (d_t \bar{x})^2 + (d_t \bar{y})^2 + (d_t \bar{z})^2 \}] + \Sigma\{mr^2 (d_t \theta)^2\}.$$

The former term of the right-hand member of this equation is the vis viva due to the velocity of the centre of gravity; and in the other term ($r_1 d_t \theta_1$) is equal to the velocity of m_1 about a parallel axis through the centre of gravity, so that $\Sigma\{m(r d_t \theta)^2\}$ is the vis viva due to the rotation about the centre of gravity.

212. $\Sigma\{mr^2 (d_t \theta)^2\} = (d_t \theta)^2 \cdot \Sigma(mr^2)$. Now $d_t \theta$ is the angular velocity of the body; and $\Sigma(mr^2)$ is called the *moment of inertia* of the body about the centre of gravity. Hence the vis viva due to rotation is equal to the moment of inertia about the centre of gravity multiplied into the square of the angular velocity.

213. If a system be acted on by no impressed forces, or by forces which always satisfy the conditions of equilibrium, the vis viva is constant.

For in both cases $\Sigma \{m(Xdx + Ydy + Zdz)\} = 0$, and therefore equation (A) of Art. 203 is reduced to

$$\Sigma(mv^2) = C.$$

From the equation of vis viva we may deduce a principle applicable to all systems to which that equation applies; viz.:

THE PRINCIPLE OF LEAST ACTION.

214. *When a system of particles either connected or unconnected, and either constrained to move on given curve lines or not, be in motion by the action of external forces which are not impulsive; the change of value of the function $\Sigma \int_1 (mv)$ in passing from one given position to another is less than if the particles had taken any other paths than those which they do take.*

Denote the above function by V ; then we are to prove that the variation of V between given limits is equal to 0. Now using the symbol δ as explained in the Calculus of Variations, we have

$$\begin{aligned} \delta V &= \delta \Sigma \int_1 (mv) = \delta \Sigma \int (mvd s) \\ &= \Sigma \delta \int (mvd s) \\ &= \Sigma \int (mv \delta ds + mds \delta v) \\ &= \Sigma \int (mvd \delta s + mvd t \delta v) \\ &= \Sigma \int (mvd_i \delta s + mvd \delta v). \end{aligned}$$

But whether a particle move on a curve line or in free space, we have

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2;$$

$$\therefore ds \cdot \delta ds = dx \cdot \delta dx + dy \cdot \delta dy + dz \cdot \delta dz,$$

or, by transposing the symbols δ and d ,

$$ds \cdot d \delta s = dx \cdot d \delta x + dy \cdot d \delta y + dz \cdot d \delta z;$$

$$\therefore v \cdot d_i \delta s = d_i x \cdot d_i \delta x + d_i y \cdot d_i \delta y + d_i z \cdot d_i \delta z,$$

and

$$\therefore \Sigma (mvd_i \delta s) = \Sigma \{m(d_i x d_i \delta x + d_i y d_i \delta y + d_i z d_i \delta z)\} \dots (1).$$

Now δV is the change of value of V in consequence of the particles being made to take new paths. But it is impossible for them to take new paths unless compelled to do so by the introduction of new forces; and by the nature of the case these additional forces are not to affect the motion except in *direction*; they must therefore at every moment act at *right angles* to the *new* paths, they will therefore (Art. 206) not appear in the equation of vis viva for the new paths.

The equation of vis viva for the original paths is

$$\Sigma(mv^2) = 2\Sigma\{m \int (Xdx + Ydy + Zdz)\} + C.$$

But as the right-hand member is an exact integral function of x, y, z (Art. 144) denote it by $2\Sigma(mU) + C$;

$$\therefore \Sigma(mv^2) = 2\Sigma(mU) + C,$$

where $d_x U = X$, $d_y U = Y$, and $d_z U = Z$.

Now since it has been shewn that the forces which were introduced to change the paths do not appear in the new equation of vis viva, we have merely to write $x + \delta x$, $y + \delta y$, $z + \delta z$ in the equation just found, in order to obtain the equation of vis viva for the new paths. Hence, by Taylor's theorem,

$$\begin{aligned} & \Sigma\{m(v + \delta v)^2\} \\ &= 2\Sigma\{m(U + d_x U \delta x + d_y U \delta y + d_z U \delta z + \dots)\} + C. \end{aligned}$$

Subtracting the former equation from this, and retaining only terms containing the first powers of the variations, we have

$$\begin{aligned} \Sigma(2mv\delta v) &= 2\Sigma\{m(d_x U \delta x + d_y U \delta y + d_z U \delta z)\}; \\ \therefore \Sigma(mv\delta v) &= \Sigma\{m(X\delta x + Y\delta y + Z\delta z)\}. \end{aligned}$$

But we know from Art. 200, that since δx , δy , δz are *possible* displacements of the first order of small quantities, the equation

$$\Sigma\{m(d_x^2 x \delta x + d_y^2 y \delta y + d_z^2 z \delta z)\} = \Sigma\{m(X\delta x + Y\delta y + Z\delta z)\},$$

holds good. (This step requires that the connection of the parts of the system shall not be broken, and that particles

which were on curves or surfaces in the first case, shall continue to move on the same curves or surfaces as before.)

$$\therefore \Sigma(mv\delta v) = \Sigma\{m(d_i^2 x \delta x + d_i^2 y \delta y + d_i^2 z \delta z)\}.$$

$$\text{But } \Sigma(mv d_i \delta s) = \Sigma\{m(d_i x d_i \delta x + d_i y d_i \delta y + d_i z d_i \delta z)\}.$$

Hence, adding these two equations together, we obtain

$$\Sigma(mv d_i \delta s + mv \delta v) = d_i \Sigma\{m(d_i x \delta x + d_i y \delta y + d_i z \delta z)\};$$

$$\therefore \delta V = \Sigma\{m(d_i x \delta x + d_i y \delta y + d_i z \delta z)\} + \text{const.}$$

Since the first and last positions of the system are given, the values of δx , δy , δz are zero at both the limits of the integral, and hence $\delta V = 0$, that is, V is either a maximum or a minimum; it cannot be a maximum, since by causing the particles to take circuitous routes we may make V as *large* as we please: consequently V is a minimum.

215. If the system be acted on by no forces, $\Sigma(mv^2)$ is constant, and hence

$$V = \int \Sigma(mv ds) = \int \Sigma(mv^2 dt) = \int (C dt) = Ct.$$

Hence Ct or t is a minimum; that is, the system will pass from one given position to another, in less time than if it were compelled to take any other route than that, which in accordance with the Laws of Motion it does actually choose for itself.

216. If there be but one particle and it be subject to the action of no force, then v is constant, and

$$V = \int \Sigma(mv ds) = mvs,$$

that is, s is a minimum; consequently the particle in passing from one given position to another, will describe the shortest line that can be drawn between them upon the given surface.

217. Remarks. We have seen that the principle of least action depends upon $Xdx + Ydy + Zdz$ being an exact differential of a function of xyz ; it is therefore not true when

sliding friction acts, or when the system moves in a resisting medium; nor, in a word, in any of the cases enumerated in the Remarks upon the principle of Vis Viva.

Since $\Sigma \int (m v ds) = \Sigma \int (m v^2 dt) = \int_t \Sigma (m v^2)$ the principle has been thus stated: *The sum of the Vires Vivæ, taken during the time that a system employs in passing from one given position to another, is less than if the particles had taken any other courses.*

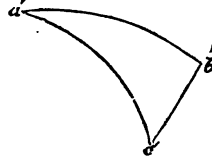
THE PRINCIPLE OF LEAST RESTRAINT.

218. In our remarks on the Principle of Vis Viva, we have shewn that in a system of bodies there are in general certain forces which disappear from the equation. These have consequently no effect upon the vis viva of the system. Notwithstanding this disappearance, however, they are not wholly without effect, but expend themselves in *modifying the paths* and *constraining the motions* of the particles on which they act; from which circumstance they are called *forces of restraint*.

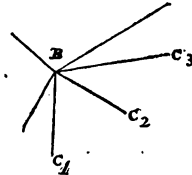
From the principle enunciated in Art. 177 we know that the forces of restraint are such as would, when applied to the system, produce equilibrium: consequently they can exert no influence upon the motion of the common centre of gravity. (Art. 196). From this circumstance we are able to deduce the general property of motion which is to form the subject of demonstration in this article. The principle may be thus enunciated.

The motion of a system of particles, connected in any manner whatever, whose motions are modified by any external restraints whatever, proceeds in every instance in the greatest possible accordance with free motion, or under the least possible restraint; that is, if the particles $m_1, m_2, m_3, \dots$ be respectively forced from their free paths through the small spaces $\sigma_1, \sigma_2, \sigma_3, \dots$ in the very small time τ , then $\Sigma(m\sigma^2)$ is a minimum.

Let a_1c_1 be the actual path of any one of the particles m_1 , and a_1b_1 the path which would have been described in the same time τ if it had been free (that is, if the forces of restraint had not acted). Join b_1c_1 ; this is the direction in which the resultant of the restraining pressures acts upon m_1 , and it is also the space through which that resultant deflects m_1 in the time τ ; consequently $b_1c_1 = \sigma_1$. Similarly $b_2c_2 = \sigma_2$, $b_3c_3 = \sigma_3$...



Let us now detach all the particles from the system and collect them in a free state at some point B in space, where let them be simultaneously acted upon for the time τ in their proper directions, by pressures equal to the respective forces of restraint. By such action m_1 , m_2 , m_3 , ... will move from B to C_1 , C_2 , C_3 , ... such that BC_1 , BC_2 , BC_3 , ... are respectively equal and parallel to b_1c_1 , b_2c_2 , b_3c_3 , ... And because, as we have shewn, the forces of restraint are such as alone would produce equilibrium upon a free rigid system, B remains still the centre of gravity of the particles when at C_1 , C_2 , C_3 , ... (Art. 154). Hence by a property of the centre of gravity



$\Sigma(m \cdot BC^2)$ is a minimum;

or $\Sigma(m\sigma^2)$ is the least possible.

When it is said that $\Sigma(m\sigma^2)$ is a minimum, it is meant, that its value is less than if the forces of restraint were not such as would keep the system in equilibrium. For it is only when these forces are such as would keep the system in equilibrium, that B remains the common centre of gravity (Art. 196) of the particles after they have been moved into the positions C_1 , C_2 , C_3 , ... Hence, *The motion, &c.*

IMPULSIVE ACTION.

219. Before concluding this chapter, it may be proper to add a few remarks on the method of applying to impulsive action the preceding Dynamical Principles. In Art. 26 it was shewn, that *impulsive* action produces a finite pressure of a

finite duration; it therefore falls under the class of finite forces, the properties of which we have investigated in the present chapter. Since, however, the duration of impulsive action is so short, that the whole effect *appears* to be produced almost instantaneously, it always happens that the object of an investigation is to discover the state of a system immediately after the impulse has *ceased* to act. On this account, we are relieved from the necessity of inquiring into the circumstances of the system during the continuance of the action; which might be obtained, were they required, from the principles we have investigated. Our object now is to shew how we may step at once, from the known circumstances immediately preceding an impulse, to the equations and properties which obtain at the moment the impulse ceases to act. To this end, we remark, that on account of the velocities being finite (Art. 26), while the duration of the action of the impulses is inappreciable, $d_x x$, $d_y y$, $d_z z$ will pass through continuous changes during the impulse, and yet x , y , z will not sensibly change in that time; we may therefore integrate the equations of motion on the supposition, that the co-ordinates of the points of the system affected by impulsive action are invariable from the beginning to the end of the impulse; and that $d_x x$, $d_y y$, $d_z z$ and their differential coefficients with regard to t are alone variable.

In Arts. 33, 34 we have seen that

$$\text{impulse} = P_1 t_1 + P_2 t_2 + P_3 t_3 + \dots = \int P dt$$

that is, the integral of a pressure with regard to t , taken during the time of its continuance is equal to the impulse, or momentum, communicated to the system.

220. It will be seen at once from the above statement that the six equations of motion for ordinary forces will hold for impulsive forces also: and consequently all the general principles which have been deduced from them in the preceding pages of this chapter, with the exception of the equation of vis viva, and those which depend upon it. The reason of this exception is that δx , δy , δz in the investigation of that principle are not arbitrary but actual displacements, and in impulsive action there are no actual displacements.

If we integrate the six equations of Art. 181 we shall have the following six equations proper to impulsive forces.

Let U_1, V_1, W_1 be the momenta communicated to the particle m_1 parallel to the axes of x, y, z . $U_2, V_2, W_2, U_3, V_3, W_3, \dots$ similar quantities for $m_2, m_3, \dots$. Then

$$\Sigma (m d_i x) = \Sigma \int_i (m X) = \Sigma (U) + C_1,$$

$$\Sigma (m d_i y) = \Sigma \int_i (m Y) = \Sigma (V) + C_2,$$

$$\Sigma (m d_i z) = \Sigma \int_i (m Z) = \Sigma (W) + C_3,$$

$$\Sigma \{m (y d_i z - z d_i y)\} = \Sigma \int_i (m Z y - m Y z) = \Sigma (W y - V z) + C_4,$$

$$\Sigma \{m (x d_i z - z d_i x)\} = \Sigma \int_i (m X z - m Z x) = \Sigma (U z - W x) + C_5,$$

$$\Sigma \{m (x d_i y - y d_i x)\} = \Sigma \int_i (m Y x - m X y) = \Sigma (V x - U y) + C_6.$$

CHAPTER VII.

ON THE MOMENT OF INERTIA.

221. MENTION was first made of the subject to which this chapter is devoted in Art. 212. The *moment of inertia* of a particle is *the product of its mass into the square of its distance from a fixed line*. A very simple theorem which we shall prove in Art. 224, enables us to deduce the moment of inertia of a body with respect to a given line, from its moment of inertia with respect to a line, drawn through the centre of gravity, parallel to the given line. On this account, it is sufficient to know the moment of inertia of a body about any one line, in order to be able to find by a short process the moment of inertia about any other line which is parallel to it, whether it pass through the centre of gravity or not. We shall therefore investigate the moment of inertia with regard to any one line which is most convenient for our investigation.

222. *To find the moment of inertia of a uniform straight line or very narrow parallelogram about an axis perpendicular to it passing through one end.*

Let AB be the given straight line,

M = its mass, $a = AB$ its length,

$AP = x$, $PQ = \delta x$.

$\left[\begin{array}{c} A \\ P \\ Q \\ B \end{array} \right.$

Then the mass of the element $PQ = M \cdot \frac{\delta x}{a}$,

and its moment of inertia $= M \cdot \frac{\delta x}{a} \cdot x^2$;

∴ the moment of inertia of the whole line

$$= \int_0^a M \frac{x^2}{a}$$

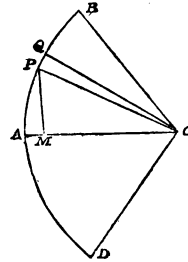
$$= \frac{M}{3} \cdot \frac{x^3}{a} + C$$

$$= M \frac{a^3}{3},$$

the limits of integration being $x = 0$, $x = a$.

223. *To find the moment of inertia of a circular arc BAD about a diameter AC which bisects it.*

Let C be the centre of the circle; PQ any small elementary portion of the arc: join PC , QC , BC , DC , and draw PM perpendicular to AC .



$$a = AC, \quad a = BCA = DCA,$$

$$M = \text{the mass of the arc.}$$

$$\theta = PCA, \quad \delta\theta = PCQ.$$

$$\text{Then the mass of } PQ = M \cdot \frac{\delta\theta}{2a},$$

$$\text{and, since } PM = a \sin \theta,$$

$$\text{mom. inert. of } PQ = M \frac{\delta\theta}{2a} \cdot a^3 \sin^2 \theta;$$

$$\therefore \text{mom. inert. of whole arc} = \frac{Ma^2}{2a} \cdot \int_{\theta} \sin^2 \theta \cdot \left\{ \begin{array}{l} \text{from } \theta = -a \\ \text{to } \theta = +a \end{array} \right\}$$

$$= \frac{Ma^2}{4a} \cdot \int_{\theta} (1 - \cos 2\theta), \text{ same limits,}$$

$$= \frac{Ma^2}{4a} \left(\theta - \frac{1}{2} \sin 2\theta \right) \dots\dots\dots$$

$$= \frac{Ma^2}{2a} \left(a - \frac{1}{2} \sin 2a \right)$$

$$= M \frac{a^3}{2} \left(1 - \frac{\sin 2a}{2a} \right).$$

224. *The moment of inertia of a body with respect to any line, is equal to its moment of inertia with respect*

to a line parallel to it through its centre of gravity, together with its moment about the original line, on the supposition of its whole mass being condensed into its centre of gravity.

For convenience, let the axis of x be taken parallel to the original line, and passing through the centre of gravity.

Let $m_1, m_2, m_3 \dots$ be the masses of the particles of the body, $x_1, y_1, x_2, y_2, x_3, y_3, \dots$ their co-ordinates, and suppose a, b , the co-ordinates of the point in which the original line cuts the plane xy . Then the squares of the distances of the particles from this line are respectively

$$(x_1 - a)^2 + (y_1 - b)^2 = x_1^2 + y_1^2 - 2ax_1 - 2by_1 + a^2 + b^2$$

$$(x_2 - a)^2 + (y_2 - b)^2 = x_2^2 + y_2^2 - 2ax_2 - 2by_2 + a^2 + b^2$$

&c.

&c.

Hence the moment of inertia about this line

$$= \Sigma \{m(x^2 + y^2)\} - 2a\Sigma(mx) - 2b\Sigma(my) + (a^2 + b^2)\Sigma m.$$

But the first term is the moment of inertia about the axis of x ; the second and third terms are zero by the nature of the centre of gravity; and the last term is the moment of inertia of the whole mass collected at the centre of gravity. Therefore, *The moment, &c.*

225. Hence we perceive that the moment of inertia with respect to a line through the centre of gravity, is less than that with respect to any other parallel line whatsoever.

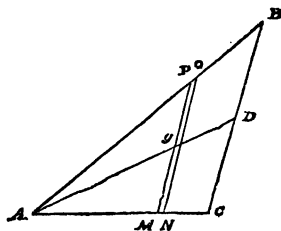
226. To find the moment of inertia of a plane triangle with respect to a line through one of its angular points perpendicular to its plane.

Let ABC be the triangle, A the angular point through which the line passes. PQ a very small part of AB : draw PM, QN parallel to BC , and let AD bisect BC .

M = the mass of the triangle,

$a = BC$, $b = AC$, $c = AB$,

$x = AM$, $\delta x = MN$,



u = the moment of inertia of the triangle AMP about A ,

δu = the moment of inertia of $PMNQ$ about A .

Then δu the moment of inertia of the elementary portion PN , which we may regard as a very narrow parallelogram, whose centre of gravity is g the middle point of MP ,

= (its mass) $(Ag)^2$ + its mom. inert. about g Art. 224.

Also by Article 222, the moment of inertia of PN about g

= moment of inertia of gN + moment of inertia of gQ

$$= \frac{\text{mass of } PN}{2} \cdot \frac{gM^2}{3} + \frac{\text{mass of } PN}{2} \cdot \frac{gP^2}{3}$$

$$= (\text{mass of } PN) \cdot \frac{Mg^2}{3}.$$

Hence $\delta u = (\text{mass of } PN) (Ag^2 + \frac{1}{3}Mg^2)$.

$$\begin{aligned} \text{But mass of } PN &= M \cdot \frac{\text{area of } PN}{\text{area of } \Delta} = M \cdot \frac{\frac{ax}{b} \cdot \delta x \cdot \sin C}{\frac{1}{2}ab \sin C} \\ &= 2M \cdot \frac{x \delta x}{b^2}, \end{aligned}$$

also $Ag = AD \cdot \frac{x}{b}$, and $Mg = \frac{1}{2} \cdot \frac{ax}{b}$;

$$\therefore \delta u = 2M \cdot \frac{x \delta x}{b^2} \left(AD^2 \cdot \frac{x^2}{b^2} + \frac{a^2 x^2}{12b^2} \right),$$

$$\text{and } \therefore d_u u = \frac{2M}{b^4} \cdot \left(AD^2 + \frac{1}{12}a^2 \right) \cdot x^3.$$

This being integrated between the limits $x = 0$, and $x = b$, gives the moment of inertia of the whole triangle

$$\begin{aligned} &= \frac{2M}{b^4} \left(AD^2 + \frac{1}{12}a^2 \right) \cdot \frac{b^4}{4} \\ &= \frac{M}{24} (a^2 + 12AD^2). \end{aligned}$$

This may be expressed, if necessary, in terms of a, b, c ;
for by the property of triangles

$$4AD^2 = 2b^2 + 2c^2 - a^2 ;$$

$$\therefore \text{moment of inertia of } \Delta = \frac{M}{24} (6b^2 + 6c^2 - a^2)$$

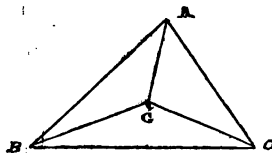
$$= \frac{M}{12} (3b^2 + 3c^2 - a^2).$$

227. *To find the moment of inertia of a triangle with respect to a line perpendicular to its plane through its centre of gravity.*

Let ABC be the triangle, G its centre of gravity ;
 M its mass ; join GA, GB, GC .

$$a = BC, \quad b = AC, \quad c = AB.$$

The lines from G divide the triangle into three equal triangles,
the mass of each of which is $\frac{M}{3}$.



Hence by the last article,

$$\text{moment of inertia of } BGC = \frac{M}{36} \cdot (3BG^2 + 3CG^2 - a^2),$$

$$\dots\dots\dots AGC = \frac{M}{36} \cdot (3AG^2 + 3CG^2 - b^2),$$

$$\dots\dots\dots AGB = \frac{M}{36} \cdot (3AG^2 + 3BG^2 - c^2) ;$$

$\therefore$ moment of inertia of ΔABC

$$= \frac{M}{36} \cdot \{6(AG^2 + BG^2 + CG^2) - (a^2 + b^2 + c^2)\}$$

$$= \frac{M}{36} \cdot \{2(a^2 + b^2 + c^2) - (a^2 + b^2 + c^2)\}$$

$$= \frac{M}{36} \cdot (a^2 + b^2 + c^2).$$

228. *To find the moment of inertia of a circular area with respect to a line perpendicular to its plane through its centre.*

Let a = the radius of circle,

M = its mass.

Divide the whole circle into very narrow annuli; and let r and $r + \delta r$ be the internal and external radii of any one of them. Then its area is ultimately equal to $2\pi r \delta r$, and therefore

$$\text{the mass of the annulus} = M \cdot \frac{2\pi r \delta r}{\pi a^2};$$

$$\therefore \text{the moment of inertia of this element} = M \cdot \frac{2r \delta r}{a^2} \cdot r^2.$$

Consequently the moment of inertia required

$$= \int_r \frac{2Mr^3}{a^2}$$

between the limits $r = 0$ and $= a$;

$$\begin{aligned} \therefore \text{moment of inertia} &= \frac{2Mr^4}{4a^2} + \text{constant} \left\{ \begin{array}{l} \text{between } r = 0 \\ \text{and } r = a \end{array} \right\} \\ &= M \cdot \frac{a^2}{2}. \end{aligned}$$

229. *The moment of inertia of any plane area with respect to a line through the origin perpendicular to its plane, is equal to the sum of the moments of inertia with respect to the axes of x and y .*

Let the figure be divided into an indefinitely large number of small masses, of which let m be any one; and x , y its co-ordinates.

Then its moment of inertia about the proposed axis

$$= m(x^2 + y^2)$$

$$= my^2 + mx^2;$$

$\therefore$ moment of inertia of whole figure $= \Sigma (my^2) + \Sigma (mx^2)$.

But

$\Sigma (my^2)$ = momentum of element with respect to axis of x ,

and

$\Sigma (mx^2)$ = ... y ,

Hence the proposition is true.

230. Hence it follows that *the sum of the moments of inertia about any two rectangular axes, in the plane of the figure, having the same origin, is the same.*

For, any two may be taken as the axes of x and y , and then the sum of the moments of inertia with respect to them, is equal to the moment with respect to a line perpendicular to the plane of the figure.

231. *To find the moment of inertia of a circular area with respect to a diameter.*

Let M = the mass of the circle, a = its radius.

Then $M \cdot \frac{a^2}{2}$ = the moment of inertia with respect to an axis through its centre perpendicular to its plane.....Art. 224.

= moment of inertia with respect to axis of x
+, y

= 2 . moment of inertia with respect to a diameter ;

$\therefore$ moment of inertia required $= M \cdot \frac{a^2}{4}$.

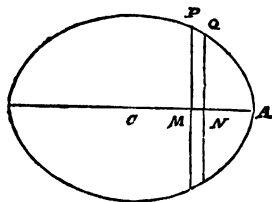
232. *To find the moment of inertia of an elliptic area with respect to the major axis.*

Let a, b be the semiaxes, $M =$
the mass of the ellipse,

$$CM = x, MN = \delta x, MP = y.$$

$$\text{Then the mass of } PN = M \cdot \frac{y \delta x}{\pi ab},$$

$$\text{its moment of inertia} = M \cdot \frac{y \delta x}{\pi ab} \cdot \frac{y^2}{3} \dots (\text{Art. 222});$$



$\therefore$ moment of inertia of half the ellipse

$$= \frac{M}{3\pi ab} \cdot \int x y^3 \left\{ \begin{array}{l} \text{from } x = -a \\ \text{to } x = +a \end{array} \right\}.$$

Now assume $x = a \sin \theta$, and therefore $y = b \cos \theta$;

$\therefore$ moment of inertia of the ellipse

$$\begin{aligned} &= \frac{2M}{3\pi ab} \cdot \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} y^3 d_{\theta} x \\ &= \frac{2M}{3\pi ab} \cdot b^3 a \cdot \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^4 \theta, \dots \dots \dots \\ &= \frac{Mb^3}{6\pi} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (1 + \cos 2\theta)^2, \dots \dots \dots \\ &= \frac{Mb^3}{6\pi} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (1 + 2 \cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta), \dots \dots \dots \\ &= \frac{Mb^3}{6\pi} \left(\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8} \sin 4\theta \right), \dots \dots \dots \\ &= M \frac{b^3}{4}. \end{aligned}$$

Similarly, the mom. inert. about the minor axis $= M \cdot \frac{a^3}{4}$.

Consequently the moment of inertia of the ellipse with respect to a line through its centre perpendicular to its plane

$$\begin{aligned}
 &= M \cdot \frac{b^2}{4} + M \cdot \frac{a^2}{4} \dots\dots (\text{Art. 231}) \\
 &= M \cdot \frac{a^2 + b^2}{4}.
 \end{aligned}$$

233. *To find the moment of inertia of a cone with respect to its axis.*

Let a = its altitude, b = the radius of its base, and M = its mass.

Take a circular slice of the cone comprehended between two planes parallel to the base at the distances x and $x + \delta x$ from the vertex.

Hence the radius of this slice = $\frac{bx}{a}$, and the volume of it

$$= \pi \left(\frac{bx}{a} \right)^2 \delta x,$$

and the volume of the whole cone = $\frac{\pi}{3} ab^2$, consequently

$$\text{the mass of the slice} = \frac{3M}{a^3} \cdot x^2 \delta x,$$

$$\text{and its moment of inertia} = \frac{3M}{a^3} x^2 \delta x \cdot \frac{b^2 x^2}{2} \dots\dots (\text{Art. 228});$$

hence the moment of inertia of the cone

$$\begin{aligned}
 &= \frac{3Mb^2}{2a^5} \cdot \int_0^a x^4 \left\{ \begin{array}{l} \text{from } x=0 \\ \text{to } x=a \end{array} \right\} \\
 &= \frac{3Mb^2}{2a^5} \cdot \frac{a^5}{5} \\
 &= M \cdot \frac{3b^2}{10}.
 \end{aligned}$$

234. *To find the moment of inertia of a sphere with respect to a diameter.*

Let this diameter be the axis of x , the centre being the origin of co-ordinates. Take a circular slice comprehended between two planes perpendicular to the diameter at the distances

x and $x + \delta x$ from the origin. The radius of this slice being y , its volume will be $\pi y^2 \delta x$,

and since the volume of the sphere = $\frac{4}{3} \pi a^3$,

$$\text{the mass of the slice} = \frac{3M}{4a^3} \cdot y^2 \delta x,$$

$$\text{and its moment of inertia} = \frac{3M}{4a^3} \cdot y^2 \delta x \cdot \frac{y^2}{2} \dots \text{Art. 228;}$$

$$\begin{aligned} \therefore \text{mom. inert. of the sphere} &= \frac{3M}{8a^3} \cdot \int_a^{-a} y^4 \dots \dots \left\{ \begin{array}{l} \text{from } x = -a \\ \text{to } x = +a \end{array} \right\} \\ &= \frac{3M}{8a^3} \int_a^{-a} (a^2 - x^2)^2 \dots \dots \dots \\ &= \frac{3M}{8a^3} \int_a^{-a} (a^4 - 2a^2x^2 + x^4) \dots \dots \\ &= \frac{3M}{8a^3} \left(a^4x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5 \right) \dots \\ &= M \cdot \frac{2a^2}{5}. \end{aligned}$$

235. From this we may find the moment of inertia of a spherical shell, the external and internal radius of which are a, b .

For let M = the mass of the shell.

Then, because similar solid figures are proportional to the cubes of their like dimensions,

mass of the external sphere : mass of the internal sphere :: $a^3 : b^3$;

$$\therefore \text{mass of external sphere} : M :: a^3 : a^3 - b^3;$$

$$\therefore \text{mass of external sphere} = \frac{Ma^3}{a^3 - b^3}.$$

$$\text{Consequently, its moment of inertia} = \frac{Ma^3}{a^3 - b^3} \cdot \frac{2a^2}{5}.$$

Similarly,

the moment of inertia of the internal sphere = $\frac{M b^3}{a^3 - b^3} \cdot \frac{2 b^2}{5}$;

$$\therefore \text{moment of inertia of the shell} = M \cdot \frac{2}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3}.$$

236. *To find the moment of inertia of an ellipsoid with respect to one of its principal axes.*

Let the mass of the ellipsoid be M and the equation of its surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and let the moment of inertia be required with respect to the axis of x .

Take a small elementary parallelopiped, of which the volume is $\delta x \delta y \delta z$; then, representing the density by μ , its mass = $\mu \delta x \delta y \delta z$; and the distance of this from the axis of x being $\sqrt{y^2 + z^2}$,

the moment of inertia of this element = $\mu \delta x \delta y \delta z (y^2 + z^2)$.

Hence the moment of inertia of the ellipsoid

$$= \mu \int_x \int_y \int_z (y^2 + z^2),$$

the limits for z being such as satisfy the above equation of the surface, they are therefore

$$-c \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} \text{ and } +c \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}:$$

the limits for x being such values as satisfy $z = 0$, or

$$1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0;$$

they are therefore

$$-a \left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}} \text{ and } +a \left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}}:$$

and the limits for y being such as to satisfy both $z = 0$

and $x = 0$, or $1 - \frac{y^2}{b^2} = 0$, which are $-b$ and $+b$.

To find $\int_x \int_y \int_z y^2$, we proceed thus;

$$\begin{aligned}
 \int_x y^2 &= y^2 x \dots \dots \left\{ \begin{array}{l} \text{from } x = -c \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{\frac{1}{2}} \\ \text{to } x = +c \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^{\frac{1}{2}} \end{array} \right\} \\
 &= 2cy^2 \left(1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} \right)^{\frac{1}{2}}; \\
 \therefore \int_x \int_x y^2 &= \frac{2cy^2}{a} \int_x \left(a^2 - \frac{a^2 y^2}{b^2} - x^2 \right)^{\frac{1}{2}} \dots \left\{ \begin{array}{l} \text{from } x = - \left(a^2 - \frac{a^2 y^2}{b^2} \right)^{\frac{1}{2}} \\ \text{to } x = + \left(a^2 - \frac{a^2 y^2}{b^2} \right)^{\frac{1}{2}} \end{array} \right\} \\
 &= \frac{c}{a} \cdot y^2 \left\{ x \left(a^2 - \frac{a^2 y^2}{b^2} - x^2 \right)^{\frac{1}{2}} \right. \\
 &\quad \left. + \left(a^2 - \frac{a^2 y^2}{b^2} \right) \sin^{-1} \frac{x}{\left(a^2 - \frac{a^2 y^2}{b^2} \right)^{\frac{1}{2}}} \right\} \dots \dots \dots \\
 &= \frac{c}{a} \cdot y^2 \cdot \left(a^2 - \frac{a^2 y^2}{b^2} \right) \pi; \\
 \therefore \int_x \int_y \int_z y^2 &= \frac{ac\pi}{b^2} \int_y (b^2 y^2 - y^4) \dots \dots \left\{ \begin{array}{l} \text{from } y = -b \\ \text{to } y = +b \end{array} \right\} \\
 &= \frac{ac\pi}{b^2} \cdot \left(\frac{b^2 y^3}{3} - \frac{y^5}{5} \right) \dots \dots \dots \\
 &= \frac{4}{15} \pi acb^2.
 \end{aligned}$$

In a similar manner we might proceed to find

$$\int_x \int_y \int_z z^2 = \frac{4}{15} \pi abc^2;$$

therefore the moment of inertia required

$$= \frac{4}{15} \mu \pi abc (b^2 + c^2).$$

$$\text{But } M = \frac{4}{3} \mu \pi abc;$$

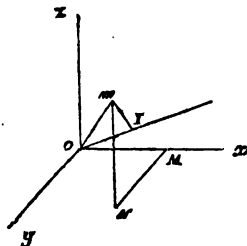
therefore the moment of inertia of the ellipsoid

$$= M \cdot \frac{b^2 + c^2}{5}.$$

GENERAL PROPERTIES.

237. To find the moment of inertia of a body with respect to any line passing through the origin of co-ordinates.

Let Ox' , Oy' , Oz' , be the three given axes, m the mass of an element whose co-ordinates are x' , y' , z' : OI the line with respect to which the moment of inertia is required, making angles α' , β' , γ' with Ox' , Oy' , Oz' , respectively. Draw mI perpendicular to OI . Then because OI is equal to the sum of projections of OM , MN , Nm upon OI ;



$$\therefore OI = x' \cos \alpha' + y' \cos \beta' + z' \cos \gamma';$$

$$\therefore (mI)^2 = (Om)^2 - (OI)^2$$

$$\begin{aligned} &= x'^2 + y'^2 + z'^2 - (x' \cos \alpha' + y' \cos \beta' + z' \cos \gamma')^2 \\ &= x'^2 \sin^2 \alpha' + y'^2 \sin^2 \beta' + z'^2 \sin^2 \gamma' - 2y'z' \cos \beta' \cos \gamma' \\ &\quad - 2x'z' \cos \alpha' \cos \gamma' - 2x'y' \cos \alpha' \cos \beta'; \end{aligned}$$

therefore the moment of inertia required

$$\begin{aligned} &= \Sigma \{m \cdot (mI)^2\} \\ &= \Sigma (mx'^2) \cdot \sin^2 \alpha' + \Sigma (my'^2) \cdot \sin^2 \beta' + \Sigma (mz'^2) \cdot \sin^2 \gamma' \\ &\quad - 2 \Sigma (my'z') \cdot \cos \beta' \cos \gamma' - 2 \Sigma (mx'z') \cdot \cos \alpha' \cos \gamma' \\ &\quad - 2 \Sigma (mx'y') \cdot \cos \alpha' \cos \beta'. \end{aligned}$$

238. We shall now proceed to shew that by referring the system to a proper set of co-ordinate axes the last three

terms of this equation may be made to vanish. Such co-ordinate axes as possess this property are called *principal axes*: the definition of which we may take to be, that they satisfy the three following equations,

$$\Sigma(myz) = 0, \quad \Sigma(mxz) = 0, \quad \Sigma(mxy) = 0.$$

If any one of the co-ordinate planes divides the body symmetrically it is evidently a *principal plane*, that is, a plane containing two of the principal axes; one principal axis will therefore be that co-ordinate axis which is perpendicular to this plane. Hence in all cases where two planes of symmetry at right angles to each other are known, we know the positions of the principal axes.

239. *Every rigid system has three principal axes at right angles to each other.*

Let us employ the figure and notation of Art. 237. If the system admit of three principal axes having the origin O , let them be Ox, Oy, Oz ; the co-ordinates of m referred to this system are, x, y, z . Let $a_1 b_1 c_1, a_2 b_2 c_2, a_3 b_3 c_3$ be the cosines of the angles which the respective axes of x, y, z , make with the original axes of $x' y' z'$. For brevity also make

$$f = \Sigma(mx'^2), \quad g = \Sigma(my'^2), \quad h = \Sigma(mz'^2).$$

$$F = \Sigma(my'z'), \quad G = \Sigma(mx'z'), \quad H = \Sigma(mx'y').$$

Then we have to shew that it is always possible to find such values of $a_1 b_1 c_1, a_2 b_2 c_2, a_3 b_3 c_3$ as shall satisfy the three equations

$$\Sigma(mxy) = 0, \quad \Sigma(mxz) = 0, \quad \Sigma(myx) = 0 \dots \dots (1).$$

Now x' is equal to the sum of the projections of x, y, z upon the axis of x' ;

$$\therefore x' = a_1 x + a_2 y + a_3 z \dots \dots \dots (2).$$

$$\text{Similarly } y' = b_1 x + b_2 y + b_3 z \dots \dots \dots (3),$$

$$z' = c_1 x + c_2 y + c_3 z \dots \dots \dots (4),$$

$$\text{and } x = a_1 x' + b_1 y' + c_1 z' \dots \dots \dots (5).$$

Multiply (5) by mx' ;

$$\begin{aligned}\therefore \Sigma(mx'x') &= a_1 \Sigma(mx'^2) + b_1 \Sigma(mx'y') + c_1 \Sigma(mx's') \\ &= a_1 f + b_1 H + c_1 G.\end{aligned}$$

Multiply (2) by mx ;

$$\begin{aligned}\therefore \Sigma(mx'x') &= a_1 \Sigma(mx'^2) + a_2 \Sigma(mx'y) + a_3 \Sigma(mx's) \\ &= a_1 \Sigma(mx'^2) \text{ from (1).}\end{aligned}$$

Hence denoting $\Sigma(mx'^2)$ by X we have

$$a_1 f + b_1 H + c_1 G = a_1 X \dots\dots\dots(6).$$

Similarly by multiplying (5) by my' , and integrating; and combining the result with the integral of $mx \times (3)$ we obtain

$$a_1 H + b_1 g + c_1 F = b_1 X \dots\dots\dots(7).$$

So from the integrals of $mx' \times (5)$ and $mx \times (4)$, we find

$$a_1 G + b_1 F + c_1 h = c_1 X \dots\dots\dots(8).$$

Finally, eliminating $a_1 b_1 c_1$ from equations (6), (7), (8), we find the following cubic equation for the determination of the value of X ,

$$\begin{aligned}(X-f)(X-g)(X-h) - F^2(X-f) - G^2(X-g) - H^2(X-h) \\ = 2FGH \dots\dots\dots(9),\end{aligned}$$

which being of odd dimensions has necessarily one real root, which being written for X in two of the equations, (6), (7), (8) will enable us, with the equation $1 = a_1^2 + b_1^2 + c_1^2$ (Art. 19) to find the values of $a_1 b_1 c_1$. Hence the position of the axis of x is determinable, and has a real existence.

Now inasmuch as equation (9) involves f, g, h, F, G, H , symmetrically, it is evident we ought to obtain the same equation for $\Sigma(my'^2)$ instead of $\Sigma(mx'^2)$ or X ; or for $\Sigma(mx'^2)$ instead of X , and by going through a similar process to that above, it is found that we do arrive at the same equation; consequently the three roots of equation (9) are the values of $\Sigma(mx'^2)$, $\Sigma(my'^2)$ and $\Sigma(mz'^2)$; these being all possible,* the

* "That the cubic equation

$$(s-a)(s-b)(s-c) - a'^2(s-a) - b'^2(s-b) - c'^2(s-c) = 2a'b'c'$$

has all its roots real, may be shewn by putting it under the form

$$(s-c)$$

positions of the axes of y and z may be determined in a manner exactly similar to that of x ; and thus a system of principal axes is shewn to exist corresponding to any proposed origin.

240. *If through the same origin there can be drawn two sets of principal axes of inertia, no axis belonging to both sets; then every set of rectangular axes having that origin are principal axes, and all the axes are axes of equal moments of inertia.*

We may here suppose the original axes Ox' , Oy' , Oz' to be principal axes, in which case $F = 0$, $G = 0$, $H = 0$, and these values being written in equations (6), (7), (8) give

$$X = f = g = h;$$

leaving $a_1b_1c_1$ indeterminate. In like manner it may be proved that $Y = f = g = h = Z$, and that $a_2b_2c_2$, $a_3b_3c_3$ are all indeterminate: hence it follows, that every system having the proposed origin is a system of principal axes; and that the moments of inertia about any one of them, and of the original axes, are the same.

241. *PROP. Given one principal axis to find the other two.*

Let the given axis be that of z , and suppose the axes of z' and z , and therefore the planes of $x'y'$ and xy , to coincide: let θ be the inclination of x to x' .

$$\begin{aligned} \text{Then } x &= x' \cos \theta + y' \sin \theta, \\ y &= -x' \sin \theta + y' \cos \theta. \end{aligned}$$

Now a condition that the co-ordinate axes of x and y may be principal axes is

$$(s-c)\{(s-a)(s-b)-c'^2\}-\{a'^2(s-a)+b'^2(s-b)+2a'b'c'\}=0,$$

and substituting for s , α and β the roots of $(s-a)(s-b)-c'^2=0$.

The results of these substitutions, since $(\alpha-a)(\alpha-b)=c'^2$, $(\alpha-\beta)(b-\beta)=c'^2$, are

$$-\{a'\sqrt{\alpha-a} \pm b'\sqrt{\alpha-b}\}^2 \text{ and } +\{a'\sqrt{\alpha-\beta} \pm b'\sqrt{b-\beta}\}^2;$$

for α is greater than both a and b , and β less. Therefore there is one root greater than α , another between α and β , and a third less than β ." (Hymers' *Geometry of Three Dimensions*, page 143, Cor. 2nd Edition.)

$$0 = \Sigma(mxy);$$

$$\begin{aligned} \therefore 0 &= \Sigma \{m(y' \sin \theta + x' \cos \theta)(y' \cos \theta - x' \sin \theta)\} \\ &= \Sigma \{m(y'^2 - x'^2)\} \sin \theta \cos \theta + \Sigma(m x' y') (\cos^2 \theta - \sin^2 \theta); \end{aligned}$$

$$\therefore \tan 2\theta = \frac{2\Sigma(m x' y')}{\Sigma \{m(x'^2 - y'^2)\}},$$

from which the value of θ is known.

242. *If through a principal axis of inertia there can be drawn more than two principal planes of inertia, then every plane through that axis is a principal plane of inertia: the axes of inertia being supposed to have a common origin: and the moments of inertia about such axes as are at right angles to the given axis are all equal.*

Using the notation and construction of last article, we find, as there, that

$$0 = \Sigma \{m(y'^2 - x'^2)\} \cdot \sin \theta \cos \theta + \Sigma(m x' y') \cdot (\cos^2 \theta - \sin^2 \theta).$$

But in this article, by hypothesis, $\Sigma(m x' y') = 0$, and neither $\sin \theta$ nor $\cos \theta$ is equal to zero, consequently θ remains indeterminate, which proves the first part of the proposition: and

$$\Sigma \{m(y'^2 - x'^2)\} = 0,$$

$$\therefore \Sigma(m x'^2) = \Sigma(m y'^2) \dots \dots \dots (1).$$

$$\begin{aligned} \text{Also } \Sigma(m x^2) &= \Sigma \{m(x' \cos \theta + y' \sin \theta)^2\} \\ &= \Sigma(m x'^2) \cdot \cos^2 \theta + \Sigma(m x' y') \cdot 2 \sin \theta \cos \theta + \Sigma(m y'^2) \cdot \sin^2 \theta \\ &= \Sigma(m x'^2) (\cos^2 \theta + \sin^2 \theta) \dots \text{from (1)} \\ &= \Sigma(m x'^2). \end{aligned}$$

$$\text{Similarly } \Sigma(m y^2) = \Sigma(m y'^2) = \Sigma(m x'^2) = \Sigma(m x^2),$$

which equations prove the last part of the proposition.

243. *Given A, B, C the moments of inertia of a body with respect to its principal axes, to determine its moment of inertia with respect to any other line having the same origin.*

Let Q be the required moment of inertia with regard to a line inclined at angles α, β, γ to the principal axes;

$$\begin{aligned}
 \therefore Q &= \Sigma (m x^2) \cdot \sin^2 \alpha + \Sigma (m y^2) \cdot \sin^2 \beta \\
 &\quad + \Sigma (m z^2) \cdot \sin^2 \gamma \dots\dots \text{Arts 237, 238.} \\
 &= \Sigma (m x^2) \cdot (\cos^2 \beta + \cos^2 \gamma) + \Sigma (m y^2) \cdot (\cos^2 \alpha + \cos^2 \gamma) \\
 &\quad + \Sigma (m z^2) \cdot (\cos^2 \alpha + \cos^2 \beta) \dots\dots \text{Art. 19.} \\
 &= \Sigma \{m (y^2 + z^2)\} \cdot \cos^2 \alpha + \Sigma \{m (x^2 + z^2)\} \cdot \cos^2 \beta \\
 &\quad + \Sigma \{m (x^2 + y^2)\} \cdot \cos^2 \gamma \\
 &= A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma.
 \end{aligned}$$

244. *The moment of inertia with respect to one of the principal axes is less, and with respect to another of them greater, than with respect to any other line having the same origin.*

$$\text{For } Q = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma.$$

$$\text{But } Q = Q \cos^2 \alpha + Q \cos^2 \beta + Q \cos^2 \gamma \dots \text{Art. 19;}$$

$$\therefore 0 = (A - Q) \cos^2 \alpha + (B - Q) \cos^2 \beta + (C - Q) \cos^2 \gamma.$$

Consequently, the three quantities $A - Q$, $B - Q$, $C - Q$ cannot be either all positive, or all negative; wherefore, at least, one is positive and one negative.

Hence if A be the greatest, and C the least of the principal moments of inertia

$$A \text{ is } > Q, \text{ and } C \text{ is } < Q;$$

and since Q is the moment of inertia with respect to any line which is not a principal axis, the proposition is proved.

245. If the origin coincide with the centre of gravity, the moment of inertia with respect to one of the principal axes is less than with respect to any other line whatsoever. Art. 225.

246. *If the moments of inertia with respect to the principal axes be equal, the moment of inertia with respect to any line having the same origin will be the same.*

For, in this case,

$$\begin{aligned}
 Q &= A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma \\
 &= A (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) \\
 &= A.
 \end{aligned}$$

247. *If the moments of inertia with respect to the three principal axes be equal, then any three rectangular axes having the same origin are principal axes.*

Let us refer the body to any three rectangular axes having the proposed origin; then (Q) the moment of inertia of the body with respect to a line, having the same origin, inclined at angles α, β, γ to the axes of x, y, z , is by Art. 237,

$$\begin{aligned} &= \Sigma (mx^2) \sin^2 \alpha + \Sigma (my^2) \sin^2 \beta + \Sigma (mz^2) \sin^2 \gamma \\ &\quad - 2 \Sigma (myx) \cos \beta \cos \gamma - 2 \Sigma (mzx) \cos \alpha \cos \gamma \\ &\quad - 2 \Sigma (mxy) \cos \alpha \cos \beta. \end{aligned}$$

But the first line of this expression is

$$\begin{aligned} &= \Sigma (mx^2) \cdot (\cos^2 \beta + \cos^2 \gamma) + \Sigma (my^2) \cdot (\cos^2 \alpha + \cos^2 \gamma) \\ &\quad + \Sigma (mz^2) \cdot (\cos^2 \alpha + \cos^2 \beta) \\ &= \Sigma \{m(y^2 + z^2)\} \cdot \cos^2 \alpha + \Sigma \{m(x^2 + z^2)\} \cdot \cos^2 \beta \\ &\quad + \Sigma \{m(x^2 + y^2)\} \cdot \cos^2 \gamma \\ &= Q \cdot \cos^2 \alpha + Q \cdot \cos^2 \beta + Q \cdot \cos^2 \gamma \dots \text{Art. 246.} \\ &= Q; \end{aligned}$$

$$\therefore Q = Q - 2 \Sigma (myx) \cos \beta \cos \gamma - 2 \Sigma (mzx) \cos \alpha \cos \gamma - 2 \Sigma (mxy) \cos \alpha \cos \beta;$$

$$\therefore 0 = \Sigma (myx) \cos \beta \cos \gamma + \Sigma (mzx) \cos \alpha \cos \gamma + \Sigma (mxy) \cos \alpha \cos \beta.$$

This equation being true for all values of α, β, γ , is equivalent to the three following,

$$\Sigma (myx) = 0, \quad \Sigma (mzx) = 0, \quad \Sigma (mxy) = 0;$$

therefore (Art. 238), the co-ordinate axes are principal axes, and as they are *any* rectangular axes having the proposed origin, the proposition is proved.

248. **PROB.** *To find the positions of all the lines, having a given origin, with respect to which the moments of inertia are equal.*

Let x, y, z be the co-ordinates of a point in any one of the lines at the distance a from the origin;

$$\therefore x^2 + y^2 + z^2 = a^2 \dots\dots\dots(1).$$

$$\text{But } A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma = Q;$$

$$\therefore A (a \cos \alpha)^2 + B (a \cos \beta)^2 + C (a \cos \gamma)^2 = a^2 Q,$$

$$\text{or, } Ax^2 + By^2 + Cz^2 = a^2 Q \dots\dots\dots(2),$$

which, as Q is constant, is the equation of an ellipsoid.

Hence the required lines form conical surfaces, having their vertices at the given origin, and having the curves of intersection of the sphere (1) and the ellipsoid (2) for their directrices.

249. *PROB. To find all the origins with respect to which the principal moments are equal.*

Let x, y, z be the co-ordinates of any particle m referred to principal axes through the centre of gravity as origin : and suppose h, k, l the co-ordinates (referred to the same axes) of any one of the origins with respect to which the principal moments are equal.

The co-ordinates of m with respect to this point are $x - h, y - k, z - l$; and, because any rectangular axes through the origin hkl are principal axes, (Art. 247);

$$\therefore 0 = \Sigma \{m(y - k)(z - l)\} = \Sigma(myz) - k\Sigma(mz) - l\Sigma(my) + kl\Sigma m.$$

$$\text{But } 0 = \Sigma(myz) \dots\dots \text{Art. 238,}$$

$$\text{and } 0 = \Sigma(mz), 0 = \Sigma(my) \text{ by nature of cent. grav.}$$

$$\therefore 0 = kl\Sigma m, \text{ or } 0 = kl.$$

$$\text{Similarly, } \therefore 0 = \Sigma \{m(x - h)(z - l)\},$$

$$\text{and } 0 = \Sigma \{m(x - h)(y - k)\};$$

$$\therefore 0 = hl \text{ and } 0 = hk.$$

Consequently, at least two of the quantities h, k, l must be equal to 0: let then $k = 0$, and $l = 0$.

To find h we proceed thus:

Let A, B, C be the moments of inertia with respect to those principal axes which have the centre of gravity for origin. Then the moments of inertia with respect to axes parallel to them, and having hkl for origin, are respectively (observing that $k = 0, l = 0$)

$A, B + (\Sigma m)h^2, C + (\Sigma m)h^2 \dots\dots\dots$ Art. 224,

which by the question are to be equal to each other ;

$$\therefore B = C,$$

$$\text{and } h^2 = \frac{A - B}{\Sigma(m)}.$$

250. From which we infer, that there can be no origin with respect to which the principal moments of inertia are equal, unless the two least of the three principal moments of inertia which have the centre of gravity for origin be equal. If the centre of gravity be an origin of equal moments of inertia, there exists no other, for then $h = 0$. If the centre of gravity be not an origin of equal moments of inertia, and the two least of the three principal moments of inertia having that point for origin be equal, then there exist two origins of equal moments of inertia, situated in the third principal axis at equal distances, on opposite sides of the centre of gravity.

CHAPTER VIII.

MOTION OF A RIGID BODY ABOUT A FIXED AXIS.

251. THE particles of a rigid body which has a *fixed* axis can only move in circles, the planes of which are at right angles to the axis, and the centres of which are in the axis. The radius of the circle which any particle describes is the perpendicular distance of the particle from the axis. The angular velocity of every particle of the body is the same, because the body is rigid.

252. *The particles of a rigid body, which has a fixed axis, being acted on by given forces; it is required to determine the motion.*

Let $m_1, m_2, m_3, \dots$ be the masses of the particles, $r_1, r_2, r_3, \dots$ their distances from the fixed axis; θ the angle through which the body has revolved in the time t ; then $d_t\theta$ is the angular velocity of every point, and therefore the linear velocity of the particles are $r_1 d_t\theta, r_2 d_t\theta, r_3 d_t\theta, \dots$ consequently the effective accelerating forces are $d_t(r_1 d_t\theta), d_t(r_2 d_t\theta), d_t(r_3 d_t\theta) \dots$ (by Art. 85.)

$$\text{or, } r_1 d_t^2\theta, r_2 d_t^2\theta, r_3 d_t^2\theta, \dots$$

Wherefore the effective moving forces are

$$m_1 r_1 d_t^2\theta, m_2 r_2 d_t^2\theta, m_3 r_3 d_t^2\theta, \dots$$

these being reversed (178) will balance the impressed forces. Now we know from *Statics*, that the only condition of equilibrium when a body has a fixed axis is, that the sum of the moments of the forces about the fixed axis is equal to zero.

Hence the moment of effective forces about a fixed axis

= moment of impressed forces about the same axis:

$$\text{or, } (m_1 r_1 d_t^2\theta) r_1 + (m_2 r_2 d_t^2\theta) r_2 + (m_3 r_3 d_t^2\theta) r_3 + \dots$$

= moment of impressed forces about the fixed axis.

But the first member of this equation

$$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots) d_t^2 \theta$$

$$= (\text{moment of inertia about fixed axis}) d_t^2 \theta;$$

$$\therefore d_t^2 \theta = \frac{\text{moment of impressed forces about fixed axis}}{\text{moment of inertia of the body about same axis}}.$$

This is a general formula for the angular accelerating force on a rigid system which has a fixed axis.

253. *To find the same as in the preceding article, when the body is acted on by impulsive forces.*

Impulsive forces produce their effects as to sense so nearly instantaneously, that ordinary finite forces, in the time that the impulses last, can produce no appreciable effect. Hence we may always calculate impulsive effects as though no other forces or motion existed.

Put F_1 for the impulsive accelerating force on the particle m_1 (in the plane of its motion), at the time t during its continuance; and let p_1 be the perpendicular from the centre of the circle, in which m_1 moves, upon the direction in which F_1 acts. Then the equation for the motion of the body at any state of the impulsive action is (by the last article), using ϕ for θ for distinction as long as the impulses last,

$$d_t^2 \phi = \frac{\Sigma (m F p)}{\Sigma (m r^2)}.$$

This, according to the principle laid down in Art. 219, being integrated with respect to t considering p constant, and the integral taken from the beginning to the end of the duration of impulsive actions, gives

$$\begin{aligned} d_t \phi &= \frac{\Sigma (p \int m F dt)}{\Sigma (m r^2)} \\ &= \frac{\Sigma (p. \text{ whole impulse communicated to } m)}{\Sigma (m r^2)} \\ &= \frac{\text{moment of all the impulses about fixed axis}}{\text{moment of inertia of the body about same axis}}. \end{aligned}$$

This is the general formula for the angular velocity *generated* by impulsive action; and it must be *added* to, or *subtracted* from, the angular velocity previously existing according as, from the circumstances of the problem, the impulse is known to be communicated in such a direction as will respectively *increase* or *diminish* the previous angular motion.

254. *The particles of a rigid body, which has a fixed axis, being acted on by given forces, not impulsive, it is required to determine the pressures on the axis.*

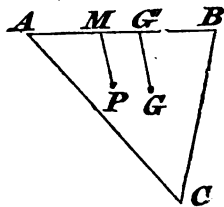
Since the body is rigid, and every part has in consequence the same angular velocity, it will be convenient to refer the particles to polar co-ordinates. Let therefore θ be the angle described about the fixed axis by the centre of gravity in the time t ; then $d_t\theta$ will be the angular velocity of every particle; and by Art. 102 the *effective* accelerating forces on m_1 are respectively $r_1 d_t^2\theta$ and $r_1 (d_t\theta)^2$ in directions of a tangent and normal to m_1 's path. These must be reversed (by Art. 178), that is, $m_1 r_1 d_t^2\theta$ must be applied to m_1 so as to *diminish* θ ; and $m_1 r_1 (d_t\theta)^2$ must be applied as a *centrifugal* pressure on m_1 . The other particles must be treated in the same way; and then these forces will balance the impressed forces, if we reckon among these latter the pressures on the axis, and consider the body free. Proceeding then on statical principles to find the conditions of equilibrium amongst these forces, we obtain equations from which we may find the pressures on the axis.

We shall illustrate what is here meant by the following problems.

255. *PROB. A triangular lamina of uniform matter oscillates about its base which is fixed in a horizontal position. Find the pressure on the axis in any position during the motion.*

Let AB be the axis about which the triangle ABC oscillates; let G be its centre of gravity, θ the angle which its plane makes with the vertical at any time t ; m the mass

of an element at any point P . Draw GG' , PM perpendicular to AB ; denote the sides by a, b, c . Take A for the origin of co-ordinates, AB the axis of x ; a line perpendicular to AB in the plane of the triangle for axis of y ; a line through A perpendicular to the plane of the triangle for axis of z , $+z$ being above the triangle.



The pressure on the axis due to the impressed force of gravity is the same at all times. Now when the plane of the triangle is vertical, this pressure ($= g \Sigma m$) may be transmitted from G to G' : at which point of the axis therefore it may be supposed always to act.

We must now calculate the pressure due to the effective forces. These forces are of two kinds; one set of parallel centrifugal pressures in the plane of the triangle parallel to $+y$. Let Y be the resultant of these, and x' the abscissa of the point in AB on which it acts;

$$\therefore Y = \Sigma \{m y (d_t \theta)^2\} = \Sigma (m y) \cdot (d_t \theta)^2 = \Sigma m \cdot \bar{y} (d_t \theta)^2;$$

$$\text{and } Y x' = \Sigma \{m y x (d_t \theta)^2\} = \Sigma (m x y) \cdot (d_t \theta)^2.$$

Now it is easily found that

$$\Sigma (m x y) = \Sigma m \cdot \bar{y} \cdot \frac{1}{4} (c + 2b \cos A);$$

$$\therefore x' = \frac{1}{4} (c + 2b \cos A).$$

The other set of pressures are $m_1 y_1 d_t^2 \theta$, $m_2 y_2 d_t^2 \theta$, ... which when reversed will act in the direction of $-z$; let Z be their resultant, and x'' the point in AB on which it acts;

$$\therefore Z = -\Sigma (m y d_t^2 \theta) = -\Sigma m \cdot \bar{y} d_t^2 \theta;$$

and taking the moments about the axis of y ,

$$\begin{aligned} Z x'' &= -\Sigma (m y x d_t^2 \theta) \\ &= -\Sigma m \cdot \bar{y} \cdot \frac{1}{4} (c + 2b \cos A) \cdot d_t^2 \theta. \end{aligned}$$

$$\therefore x'' = x'.$$

It appears then that, to know the values of Y and Z , we must calculate the values of $(d_t \theta)^2$ and $d_t^2 \theta$. These may be found from Art. 252; by which we obtain

$$\begin{aligned}
 d_t^2 \theta &= - \frac{\Sigma m \cdot g \bar{y} \sin \theta}{\Sigma (m y^2)} \\
 &= - \frac{\Sigma m \cdot g \cdot \frac{1}{3} b \sin A \cdot \sin \theta}{\frac{1}{3} \Sigma m \cdot b^2 \sin^2 A} \\
 &= - \frac{2g \sin \theta}{b \sin A}.
 \end{aligned}$$

Multiply this by $2d_t \theta$, and integrate,

$$\therefore (d_t \theta)^2 = \frac{2g}{b \sin a} (\cos \theta - \cos a),$$

a being the greatest value of θ .

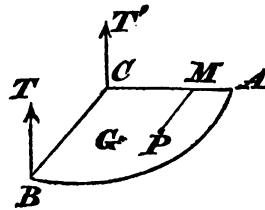
$$\text{Hence } Y = \Sigma m \cdot \frac{2}{3} g (\cos \theta - \cos a),$$

$$\text{and } Z = \Sigma m \cdot \frac{2}{3} g \sin \theta.$$

These two forces we have already shewn act on the same point of the axis; and may therefore be readily reduced if necessary to a single force acting at that point. This force, with $\Sigma m \cdot g$ acting vertically downwards at G' represent the pressures on the axis.

256. *PROB. A uniform lamina of matter in the form of a quadrant of a circle is suspended in a horizontal position by vertical strings fastened to its corners A, B, C. The string at A is cut, find the initial tensions of the other strings.*

Let a = the radius of the quadrant, G its centre of gravity, θ = the indefinitely small angle through which the plane of the quadrant has revolved about BC fixed in the time t . Let m be a small element of mass at P , $x = CM$, $y = MP$, T = tension of the string at B , T' that at C . The co-ordinates of G are $\frac{4a}{3\pi}$,



$$\frac{4a}{3\pi}.$$

Now since initially $d_t\theta = 0$, there is no initial centrifugal force: consequently the only effective force on m is $mx d_t^2\theta$ to be applied upwards. We may take the impressed forces into account by supposing the force $(\Sigma m)g$ to act at G . Now by Art. 178, there is equilibrium between the force $\Sigma m \cdot g$ at G , T at B , T' at C , and $m_1 x_1 d_t^2\theta$, $m_2 x_2 d_t^2\theta$, ... All these form a system of parallel forces, the conditions of equilibrium of which are three, viz.

$$0 = T + T' + \Sigma (mx) d_t^2\theta - \Sigma m \cdot g,$$

by resolving vertically ;

$$0 = \Sigma mg \cdot \frac{4a}{3\pi} - \Sigma (mxy) \cdot d_t^2\theta - Ta,$$

by taking moments about CA ,

$$\text{and } 0 = \Sigma mg \cdot \frac{4a}{3\pi} - \Sigma (mx^2) \cdot d_t^2\theta,$$

by taking moments about CB .

Now $\Sigma (mx) = \Sigma m \cdot \frac{4a}{3\pi}$, and $\Sigma (mx^2) =$ moment of inertia

about $BC = \Sigma m \cdot \frac{a^2}{4}$: and $\Sigma (mxy)$ may be found thus,

$$m = \Sigma m \cdot \frac{\delta x \delta y}{\frac{1}{4}\pi a^2};$$

$$\therefore \Sigma (mxy) = \frac{4\Sigma m}{\pi a^2} \cdot \iint (xy dx dy).$$

But $\iint xy dx dy = \frac{1}{2} \int x dx (y^2 + C)$ from $y^2 = 0$, to $y^2 = a^2 - x^2$

$$= \frac{1}{2} \int x dx (a^2 - x^2)$$

$$= \frac{1}{4} (a^2 x^2 - \frac{1}{2} x^4) + C \text{ from } x = 0 \text{ to } x = a$$

$$= \frac{a^4}{8};$$

$$\therefore \Sigma (mxy) = \Sigma m \cdot \frac{a^2}{2\pi}.$$

Hence by substitution, the third equation gives

$$d_t^2 \theta = \frac{\Sigma m \cdot g \cdot \frac{4a}{3\pi}}{\Sigma m \cdot \frac{a^2}{4}} = \frac{16g}{3a\pi};$$

and the second equation gives

$$T = \Sigma m \frac{4g}{3\pi} - \Sigma m \cdot \frac{a}{2\pi} \cdot \frac{16g}{3a\pi} = \Sigma m \cdot \frac{4g}{3\pi} \left(1 - \frac{2}{\pi}\right);$$

and the first gives

$$T' = \Sigma mg - \Sigma m \cdot \frac{4a}{3\pi} \cdot \frac{16g}{3a\pi} - T = \Sigma mg \cdot \left(1 - \frac{4}{3\pi} - \frac{40}{9\pi^2}\right).$$

257. *A rigid body revolves about a fixed axis, and is acted on by no impressed forces; it is required to find under what conditions there will be no pressure on the axis.*

Take the fixed axis for that of z , a perpendicular upon it through the centre of gravity of the body for that of x ; and the rest of the notation as in Art. 254. Since the body is not acted on by any impressed forces

$$d_t^2 \theta = 0, \text{ from Art. 252;}$$

$$\therefore d_t \theta = \text{constant, by integration.}$$

Hence the effective force on m_1 is $m_1 r_1 (d_t \theta)^2$ only, which is to be applied as a centrifugal force: resolve it parallel to x and y , then we have

$$m_1 x_1 (d_t \theta)^2 \text{ parallel to } x,$$

$$\text{and } m_1 y_1 (d_t \theta)^2 \text{ parallel to } y.$$

There are similar forces on $m_2, m_3, \dots$ These must satisfy the conditions of statical equilibrium of a free body, for since there is no pressure on the axis by hypothesis, the axis might be removed and the body considered free; hence

$$\Sigma (mx) (d_t \theta)^2 = 0, \text{ forces parallel to } x,$$

$$\Sigma (my) (d_t \theta)^2 = 0, \dots\dots\dots y,$$

$$\Sigma(myx)(d_t\theta)^2 = 0, \text{ moment about axis of } x,$$

$$\Sigma(mxz)(d_t\theta)^2 = 0, \dots\dots\dots y.$$

The forces have no moment about axis of x , because being centrifugal forces their directions pass through that axis. Now because $(d_t\theta)^2$ is constant, and cannot be zero by the nature of the question, the above equations furnish the following conditions,

$$\Sigma(mx) = 0, \quad \Sigma(my) = 0, \quad \Sigma(myx) = 0, \quad \Sigma(mxz) = 0;$$

or, instead of the first and second, $\bar{x} = 0, \bar{y} = 0$. Consequently the conditions that there may be no pressure on the axis, when a body in motion is acted on by no impressed forces, are,

1. *The fixed axis must pass through the centre of gravity, and*
2. *It must be a principal axis of inertia.*

258. If there be no pressure on the axis no force is requisite to restrain it in its place; it might therefore be free and the body would still continue to revolve about it with a uniform velocity. On this account the principal axes of inertia which pass through the centre of gravity have sometimes been called *permanent axes of rotation*.

259. After the explanation of impulsive action given in Art. 219 it is hardly necessary to remark that the pressure on a fixed axis, arising from impulsive agency on any part of a body, may be found exactly on the same principle as finite pressure was determined in Art. 254.

There is however one circumstance which must be carefully noted. Impulsive action will produce a sudden alteration of angular velocity, and therefore a sudden alteration of centrifugal pressure: but this change is not to be accounted an impulsive pressure. If a body be laid upon a table, a sudden increase or decrease of the force of gravity would cause the pressure of the body on the table to undergo a sudden alteration, but it would not amount to a *blow*, which is the idea attached to an impulsive pressure. In fact there cannot be impulsive pressure in any direction unless there be a sudden

finite change of velocity in that direction. This will be at once evident by consulting Art. 34. Now no change of angular velocity, however sudden, can produce a finite change of the *radial* velocity of a particle, (*i. e.* the velocity in the line in which centrifugal force acts). Consequently, there is no such thing as impulsive centrifugal pressure.

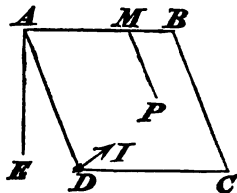
260. In the short time that an impulse lasts, finite impressed forces, such as gravity, cannot produce any appreciable effect; neither can they modify or affect in any way the effect of impulsive action. It is therefore allowable, as a mode of simplifying our investigation, in calculating impulsive effects, to neglect all other motions and forces by which the body is affected. The motion due to impulsive actions alone being thus found must be added with its proper algebraic sign to that which previously existed; the result will represent the state of the body at the end of the impulsive action.

261. *PROB. The particles of a rigid body, which has a fixed axis, being acted on by given impulses, it is required to determine the impulsive pressure sustained by the axis.*

It is evident the hypothesis of no change of position during the continuance of impulsive action amounts to the supposition that the whole impulse is communicated instantaneously. Instead therefore of using $m_1 r_1 d_t^2 \phi$ as the effective pressure on m_1 at any instant, we may use the *whole* pressure exerted during the continuance of the impulse, that is, we must employ $\int (m_1 r_1 d_t^2 \phi)$ (Art. 219), or $m_1 r_1 d_t \phi$ as the effective impulse on m_1 ; and so of $m_2, m_3, \dots$ and therefore $m_1 r_1 d_t \phi, m_2 r_2 d_t \phi, \dots$ applied to diminish ϕ will balance the impressed impulses. We may find the impulsive pressures on the axis by reckoning them among the impressed impulses, and then considering the body as free. ϕ is here used as explained in Art. 253. An example will explain the method here proposed.

262. *AB one side of a rectangular lamina ABCD of matter is fixed in a horizontal position. The figure in oscillating about AB strikes against a fixed point at D. To find the impulse on the axis.*

From A draw AE vertical; put $AB = a$, $AD = b$, Σm = mass of the figure, I the impulse at D by which it is suddenly reduced to rest: θ = inclination of the line AD to AE at the time t , previous to the impulse. Take AB , AD for axes of x , the axis of y being parallel to DI and on the same side of the figure.



We must first find the angular motion immediately previous to the impulse. By Art. 252 we have for this purpose,

$$d_t^2 \theta = - \frac{\Sigma mg \cdot \frac{1}{2} b \sin \theta}{\Sigma m \cdot \frac{1}{3} b^2} = - \frac{3g}{2b} \sin \theta :$$

the negative sign is used because gravity acts to diminish θ .

$$\therefore (d_t \theta)^2 = \frac{3g}{b} \cdot (\cos \theta + C).$$

Let now β be the value of θ when the motion began; then when $\theta = \beta$, $d_t \theta = 0$, $\therefore C = -\cos \beta$, and

$$(d_t \theta)^2 = \frac{3g}{b} (\cos \theta - \cos \beta).$$

And writing α for θ we have the angular velocity with which the plane strikes the fixed point D , viz.

$$(d_t \theta)^2 = \frac{3g}{b} (\cos \alpha - \cos \beta) \dots (1).$$

To find the effect of the blow I at D , we must apply to the particle m_1 the pressure $m_1 x_1 d_t \phi$ parallel to the axis of y so as to diminish ϕ , the angle that would be described in consequence of the impulse I : that is, we must apply $m_1 x_1 d_t \phi$ in the direction $-y$. So also $m_2 x_2 d_t \phi$, $m_3 x_3 d_t \phi$, ... These forces together with I constitute a system of parallel forces; hence the pressure on the axis must be parallel to them; denote it by Y ; and let it act at a point whose ordinate is x' . Then $-m_1 x_1 d_t \phi$, $-m_2 x_2 d_t \phi$, ..., I , Y are a system of balancing forces; and the conditions of equilibrium are

$$0 = -\Sigma (mx) d_t\phi + I + Y, \text{ forces parallel to } y,$$

$$0 = -\Sigma (m x \varkappa) d_t\phi + Y \varkappa', \text{ moments about axis of } x,$$

$$0 = -\Sigma (m x^2) d_t\phi + I b, \text{ moments about axis of } x.$$

Now

$$\Sigma (mx) = \Sigma m \cdot \frac{1}{2} b, \quad \Sigma (m x \varkappa) = \Sigma m \cdot \frac{1}{4} a b, \quad \Sigma (m x^2) = \Sigma m \cdot \frac{1}{3} b^2;$$

also, since the plane is reduced to rest by the impulse I , the value of $d_t\phi$ the angular velocity generated by I must be exactly equal to $d_t\theta$ in equation (1);

$$\therefore I = \frac{\Sigma (m x^2)}{b} d_t\phi = \Sigma m \cdot \left\{ \frac{b g}{3} \cdot (\cos \alpha - \cos \beta) \right\}^{\frac{1}{2}} \dots (2),$$

$$Y = \Sigma (mx) \cdot d_t\phi - I = \frac{1}{2} \Sigma m \cdot \left\{ \frac{b g}{3} \cdot (\cos \alpha - \cos \beta) \right\}^{\frac{1}{2}} \dots (3),$$

$$\varkappa' = \frac{\Sigma (m x \varkappa)}{Y} d_t\phi = \frac{3}{2} a \dots \dots \dots (4).$$

Equation (2) gives the magnitude of the blow at D ; (3) shews that the blow on the axis is one half of that sustained by the plane at D , and in the same direction; and (4) shews that the blow on the axis may be supposed to be applied at the distance $\frac{3}{2} AB$ from A .

263. *A rigid body having a fixed axis is struck by an impulsive force at a given point in a given direction, it is required to determine under what conditions there will be no impulsive pressure on the axis.*

As stated in Art. 260 in this problem, the body may be either at rest or in motion previous to the impulse.

It is a necessary condition that the impulse shall be communicated in a direction that coincides with a plane perpendicular to the fixed axis; for if not, it would have a component parallel to the axis, which would impel the axis in the direction of its length. Take then the fixed axis for axis of x , and the plane which contains the direction of the impulse for xy , the axis of y being parallel to and in direction of the impulse, which call Y . Since a force produces the same effect

at whatever point in the line of its direction it acts, we may suppose Y to act at a point in the axis of x , let its abscissa be x' .

The forces which balance each other are Y and $m_1 r_1 d_t \phi$, $m_2 r_2 d_t \phi$... (for there is no pressure on the axis). Resolve the latter forces parallel to x and y , then the balancing forces are

$$m_1 y_1 d_t \phi, \quad m_2 y_2 d_t \phi, \dots \text{ parallel to } x,$$

$$\text{and } Y, \quad -m_1 x_1 d_t \phi, \quad -m_2 x_2 d_t \phi, \dots \dots \dots y.$$

And therefore the conditions of equilibrium are

$$(1) \dots 0 = \Sigma (m y) d_t \phi, \text{ forces parallel to } x,$$

$$(2) \dots 0 = Y - \Sigma (m x) d_t \phi, \quad \dots \dots \dots y,$$

$$(3) \dots 0 = -\Sigma (m x y) d_t \phi, \text{ moments about axis of } x,$$

$$(4) \dots 0 = \Sigma (m y x) d_t \phi, \quad \dots \dots \dots y,$$

$$(5) \dots 0 = Y x' - \Sigma (m x^2) d_t \phi, \quad \dots \dots \dots x.$$

Equation (1) shews that $\bar{y} = 0$, and the centre of gravity is in the plane xx ; and consequently a plane drawn through the centre of gravity perpendicular to the direction of the impulse must contain the axis of rotation.

Equations (3) and (4) shew that the axis of rotation must be a principal axis of inertia.

And if $d_t \phi$ be eliminated between equations (2) (5), we have

$$x' = \frac{\Sigma (m x^2)}{\Sigma (m x)} :$$

This gives the point at which the impulse must be applied. It will be shewn that if the body were made to oscillate about the fixed axis by the action of gravity, the point here determined for the place of impact is the centre of oscillation; in reference to the question in hand, it is called the *centre of percussion*.

264. Since there is no pressure on the axis, it is not necessary that the body should have a fixed axis; and if it were struck by a blow, it would spontaneously begin to revolve about the line which in the preceding article was the axis of x ; the posi-

tion of which would be determined thus. From the centre of gravity draw a line perpendicular to the line of impulsive action: from the foot of this perpendicular draw at right angles to both these lines an axis, and make the body oscillate about that axis;—a line parallel to this through the centre of oscillation is the axis of spontaneous rotation.

265. *Origin of the terms “vis inertia” and “moment of inertia.”*

Since a body at rest cannot be moved, and a body in motion cannot have its motion changed without the expenditure of some force for that purpose, it is usual to say that matter *resists* the communication of motion. If in the word ‘*resist*’ as thus used any idea of matter’s exerting a force be implied, it certainly conveys an erroneous notion; for matter is perfectly passive; and if a force act upon it motion ensues, however small the force. The force which acts on matter, if unresisted by other forces applied also to the same matter, will be wholly *unresisted*, and its proper unresisted effect will be produced. The term *vis inertiae* (or briefly, *inertia*) has been employed to denote the fact above alluded to,—viz. that the production or change of motion can be accomplished only by *expenditure* of force: and the *vis inertia* of a given mass is evidently measured, according to this notion, by the whole quantity of force expended in communicating a given velocity to the body.

By reference to Arts. 252, 253, the reader will perceive, that if an additional particle m' were attached to a system, in angular motion, at the distance r' from the axis of motion, the effective angular acceleration would be diminished by the addition of the term $m'r'^2$ in the denominator of the fraction by which the effective angular accelerating force is expressed. On this account $m'r'^2$ is taken as the measure of the effect of the *inertia* of m' upon the angular motion of the system. To distinguish this effect from the effect of inertia on a motion of translation, it has been named the *moment of inertia*, and the whole moment of inertia of a system is measured by the sum of the products of every particle into the square of its distance from the axis of motion.

266. DEF. The *radius of gyration* is the distance from the axis at which if the whole mass were collected, the moment of inertia would be the same as before.

It is usual to denote the radius of gyration by the symbol k .

By the definition

$$(\Sigma m)k^2 = \Sigma(mr^2);$$

$$\therefore k^2 = \frac{\Sigma(mr^2)}{\Sigma m}$$

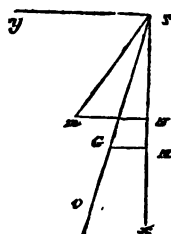
$$= \frac{\text{moment of inertia}}{\text{mass}}.$$

THE CENTRE OF OSCILLATION.

267. DEF. In Article 163 we have seen that the time of oscillation of a single particle (acted on by gravity) about a given point or axis depends upon the length of the string by which the particle is suspended; that is, upon the distance of the particle from the axis about which it oscillates. When a rigid body of finite magnitude is made to oscillate, the particles of which it is composed endeavour to complete their oscillations in different times, those which are near the axis of suspension having a tendency to oscillate quicker than those which are more remote. By reason of the rigidity of the body, they are all compelled to oscillate in the *same* time, which will evidently be intermediate to the times in which the extreme particles tend to oscillate: there is therefore some distance from the axis at which a single particle would oscillate freely in the same time as the finite body. This distance is called the *length of the simple pendulum*, the finite body being called a *compound pendulum*: and that point in a compound pendulum at which the whole body might be collected without altering the time or manner of its oscillation, is called the *centre of oscillation*.

268. PROB. A rigid body being suspended from a fixed axis; it is required to find the length of the simple pendulum.

Let the plane of the paper pass through G the centre of gravity of the body, and cut the axis of suspension perpendicularly in S : from which point draw Sx vertical and Sy horizontal, which take for the axes of co-ordinates. To them let the body be referred at the time t , when the co-ordinates of any particle m_1 are $x_1y_1z_1$.



Employing the notation of Art. 252, θ denoting the angle GSx , the effective forces are

$$m_1r_1d_t^2\theta, \quad m_2r_2d_t^2\theta, \quad m_3r_3d_t^2\theta \dots$$

these are to be applied so as to diminish θ .

The impressed forces are

$$m_1g, \quad m_2g, \quad m_3g, \dots \text{Art. 23,}$$

vertically downwards. These two sets of forces balance (177); hence taking their moments about the fixed axis, which is the only condition of equilibrium, we have

$$0 = m_1r_1^2d_t^2\theta + m_2r_2^2d_t^2\theta + m_3r_3^2d_t^2\theta + \dots$$

$$+ m_1gy_1 + m_2gy_2 + m_3gy_3 + \dots$$

$$\therefore d_t^2\theta = - \frac{g \sum (my)}{\sum (mr^2)}$$

$$= - \frac{g \bar{y} \sum m}{\sum (mr^2)}$$

$$= - \frac{g \cdot SG \cdot (\sum m) \cdot \sin \theta}{\sum (mr^2)}.$$

But if the whole body were collected at O such that $OSx = \theta'$, the same formula gives

$$d_t^2\theta' = - \frac{g \cdot SO \cdot (\sum m) \cdot \sin \theta'}{(\sum m) \cdot SO^2}.$$

By the definition $d_t^2\theta$, and $d_t^2\theta'$ are *always* equal;

$$\therefore \frac{SG \cdot (\sum m) \sin \theta}{(\sum mr^2)} = \frac{\sin \theta'}{SO}.$$

But since θ and θ' are quantities which are continually changing in value, this equation cannot always be true under the circumstances of the problem, unless both

$$\theta = \theta',$$

$$\text{and } \frac{\Sigma(mr^2)}{SG \cdot (\Sigma m)} = SO.$$

The former shews that the centre of oscillation is in a plane passing through the axis of suspension and the line SG , and the latter that its distance from S the point of suspension is equal to

$$\frac{\text{the moment of inertia about the fixed axis}}{\text{the moment of the mass about the same axis}}.$$

269. *The centres of oscillation and suspension are convertible*; that is, if the body be made to oscillate about an axis through O instead of S , the length of the simple pendulum will be unchanged, and S will be the centre of oscillation.

For if $(\Sigma m)k^2$ be the moment of inertia of the body with respect to a line through G parallel to the axis of suspension,

$$SO = \frac{\text{moment of inertia about } S}{\text{moment of mass about } S}$$

$$= \frac{(\Sigma m) \cdot k^2 + (\Sigma m) \cdot SG^2}{(\Sigma m) \cdot SG} \dots\dots\dots \text{Art 224}$$

$$= \frac{k^2}{SG} + SG \dots\dots\dots (1);$$

$$\therefore OG = \frac{k^2}{SG},$$

$$\text{and } SG = \frac{k^2}{OG};$$

$$\therefore SO = SG + GO = \frac{k^2}{OG} + OG$$

$$= \frac{(\Sigma m) \cdot k^2 + (\Sigma m) \cdot OG^2}{(\Sigma m) \cdot OG}$$

$$= \frac{\text{moment of inertia about } O}{\text{moment of mass about } O} \dots\dots \text{Art. 224,}$$

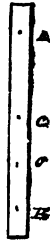
= length of the simple pendulum when the axis of suspension passes through O .

Consequently S is the centre of oscillation, and the length of the simple pendulum is the same whether the axis passes through S or through O .

270. From equation (1) it appears that the centre of oscillation is farther from the axis of suspension than the centre of gravity: for SO is greater than SG by the quantity $\frac{k^2}{SG}$.

271. The property of compound pendulums just proved has been ingeniously employed by Captain Kater in determining the length of the *second's pendulum*. The following is a brief outline of the method employed.

Let G be the centre of gravity of a pendulum, furnished with two parallel cylindrical axes of equal diameters, whose centres A, B are in a line passing through G . The body can be suspended by either of these axes and made to oscillate. By means of a sliding weight C , the times of oscillation about the two axes are made nearly equal. The oscillations are very small, and the axis from which the body is suspended rests on two perfectly horizontal planes, and does not slide but rolls. In the result, allowance is made for the buoyancy and resistance of the air.



The number of oscillations when the body is suspended from the axis A in a *day* is observed; and also when it is suspended from the axis B . These having been made nearly equal by means of the weight C , are now made exactly equal by filing away small portions from the end of the pendulum. It has, however, been found more convenient in practice, for the pendulum to rest on knife edges instead of cylindrical axes. When the numbers of oscillations in 24 hours have

been made exactly equal about both axes, let n be the number of oscillations completed in the time T , (measured in seconds)

$$\therefore \text{time of one oscillation} = \frac{T}{n},$$

and if l be the distance between the knife edges, which we have shewn to be the length of the simple pendulum, (Art. 269),

$$\frac{T}{n} = \pi \sqrt{\frac{l}{g}} \dots\dots \text{Art. 74.}$$

But if x be the length of the second's pendulum,

$$1 = \pi \sqrt{\frac{x}{g}};$$

$$\therefore \frac{T}{n} = \sqrt{\frac{l}{x}};$$

$$\therefore x = \frac{n^2 l}{T^2}.$$

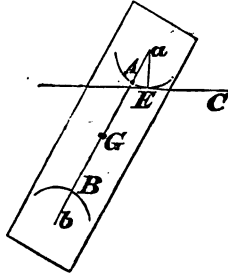
The same experiment gives the value of g

$$= \frac{n^2 \pi^2 l}{T^2}.$$

This pendulum, once adjusted, will thus enable us to determine with great accuracy the value of g at different places. By this means it has been found, that in a given latitude, g is constant for *small* altitudes; but decreases as we ascend through greater altitudes, or into higher latitudes.

272. We have mentioned in the preceding article that knife edges were found to be more convenient than cylinders for axes of suspension. We shall show however that if cylindrical axes be used, the length of the simple pendulum is equal to the distance between the cylinders, when the oscillations are very small.

Let a, b be the centres of the two cylindrical axes; G , which must be in the line ab , the centre of gravity of the pendulum. First, let it be suspended by placing it with the axis AE on the fixed horizontal plane CE . We have before stated that in oscillating the cylinder AE rolls upon the plane. Let θ be the inclination of ab to the vertical; then as an approximation sufficiently near for our present purpose (we shall afterwards prove the supposition to be true for small oscillations) we may consider E as a temporary fixed point of suspension; and therefore, applying Art. 252,



$$\begin{aligned} d_t^2 \theta &= - \frac{\Sigma m g \cdot a G \sin \theta}{\text{moment of inertia about } E} \\ &= - \frac{\Sigma m g \cdot a G \cdot \sin \theta}{\Sigma m (k^2 + GE^2)} \\ &= - \frac{g \cdot a G \cdot \theta}{k^2 + AG^2}, \text{ very nearly,} \end{aligned}$$

k being the radius of gyration about G ,

$$\therefore d_t^2 \theta + \frac{g \cdot a G}{k^2 + AG^2} \theta = 0.$$

This is the equation of vibration (Art. 152), and therefore the time of a vibration

$$= \pi \left(\frac{k^2 + AG^2}{g \cdot a G} \right)^{\frac{1}{2}};$$

or, if we denote the length of the simple pendulum which will vibrate in this time by l , we have

$$l = \frac{k^2 + AG^2}{a G}.$$

In like manner when the body is suspended from the other axis upon the plane CE , if l' be the length of the simple pendulum,

$$l' = \frac{k^2 + BG^2}{bG}.$$

But $l = l'$ by construction of the pendulum ;

$$\therefore l(aG - bG) = k^2 + AG^2 - (k^2 + BG^2) :$$

or since $aA = bB$,

$$l(AG - BG) = AG^2 - BG^2,$$

$$\therefore l = AG + BG = AB.$$

273. Remark. The result just found is simple, and for small oscillations sufficiently accurate; but it has been obtained upon several suppositions which are only approximately true, and which it is necessary to point out. We have assumed that E is a fixed point while the body describes an indefinitely small angle; this is erroneous, for the point of contact E moves with a velocity which has a constant ratio to the angular velocity of the body. We have assumed that the only effective force upon each particle is that which arises from angular motion about E , but it is evident that the point a has no angular motion; for its motion is one of translation only, being in the direction EC as θ increases. The proper mode of solving the problem is this. The motion of every particle is compounded, of an angular motion about a , and a horizontal motion of translation equal to that of a .

The corresponding effective forces on a particle m_1 whose distance from a is r_1 are, putting $Aa = c$,

$$m_1 r_1 d_t^2 \theta, \quad m_1 r_1 (d_t \theta)^2, \quad \text{for angular motion about } a \text{ (Art. 102),}$$

$$\text{and } m_1 c d_t^2 \theta, \quad \text{for the motion of translation.}$$

The first of these must be applied to m_1 so as to *diminish* θ ; the next as a centrifugal pressure tending from a ; and the last in a direction opposite to the motion of a when θ increases, that is, in the direction CE . These sets of forces balance Σmg acting vertically downwards at G , the friction F at E , and the reaction of the plane at E vertically upwards. To avoid the last, take the moments of the forces about a , and resolve them horizontally,

$$\begin{aligned}\therefore 0 &= \Sigma mg \cdot a G \sin \theta + \Sigma (mr^2) d_t^2 \theta \\ &\quad - \Sigma \{mcr \cos (\psi + \theta)\} d_t^2 \theta + Fc, \\ \text{and } 0 &= \Sigma \{mr \cos (\psi + \theta)\} \cdot d_t^2 \theta - \Sigma \{mr \sin (\psi + \theta)\} (d_t \theta)^2 \\ &\quad - \Sigma (mc) d_t^2 \theta + F,\end{aligned}$$

ψ_1 being the angle which r_1 makes with ab .

Now

$$\begin{aligned}\Sigma \{mr \cos (\psi + \theta)\} &= \cos \theta \cdot \Sigma (mr \cos \psi) - \sin \theta \Sigma (mr \sin \psi) \\ &= \cos \theta \cdot \Sigma m \cdot a G, \text{ by the nature of the centre of gravity;} \\ \text{similarly } \Sigma \{mr \sin (\psi + \theta)\} &= \sin \theta \cdot \Sigma m \cdot a G; \\ \text{also } \Sigma (mr^2) &= \Sigma m \cdot (k^2 + Ga^2); \end{aligned}$$

$$\begin{aligned}\therefore 0 &= \Sigma mg \cdot a G \sin \theta + \Sigma m (k^2 + Ga^2) d_t^2 \theta \\ &\quad - \Sigma m \cdot Aa \cdot a G \cdot \cos \theta d_t^2 \theta + F \cdot Aa, \\ \text{and } 0 &= \Sigma m \cdot a G \cos \theta d_t^2 \theta - \Sigma m \cdot a G \sin \theta (d_t \theta)^2 \\ &\quad - \Sigma m \cdot Aa d_t^2 \theta + F.\end{aligned}$$

Eliminating F between these equations, we have

$$\begin{aligned}0 &= g \cdot a G \sin \theta + (k^2 + Ga^2 + Aa^2 - 2Aa \cdot a G \cos \theta) d_t^2 \theta \\ &\quad + Aa \cdot a G \sin \theta (d_t \theta)^2 \\ &= g \cdot a G \sin \theta + \left(k^2 + AG^2 + 4Aa \cdot a G \sin^2 \frac{\theta}{2} \right) d_t^2 \theta \\ &\quad + Aa \cdot a G \sin \theta (d_t \theta)^2.\end{aligned}$$

This equation coincides with the approximate equation which was employed in the last article, if we neglect quantities of the *third* order of smallness, viz. those multiplied by $\sin^2 \frac{\theta}{2} d_t^2 \theta$, and $\sin \theta (d_t \theta)^2$, each of which is of the order θ^3 .

CHAPTER IX.

MOTION OF A RIGID BODY ABOUT A FIXED POINT.

IN preceding pages we have considered the motion of a rigid body about a fixed axis; it now only remains to determine the motion about a fixed point. This will include, as a particular case, the angular motion of a *free* body about its centre of gravity; it having been shewn in Arts. 196, 197, that the motion of a free body about its centre of gravity is the same as if that point were fixed. This chapter then will complete the determination of the motion of a rigid body.

274. **DEFS.** When that extremity of the axis, about which a body rotates, is turned towards the spectator, which makes the body appear to revolve in the direction of the hands of a watch, the angular velocity is considered to be *positive*; and it is here proposed to call that extremity of the axis the *positive* pole of rotation: and the opposite extremity about which the rotation is in the contrary direction to the hands of a watch and therefore negative, is the *negative* pole of rotation. As an example, we may mention that according to this definition, the south pole of the Earth is its positive pole of rotation, and the north pole its negative pole of rotation.

When we speak of the *inclination* of two axes of rotation, we mean the angle contained by the *positive* portions of those axes; or if the body be referred to a sphere having its centre in the fixed point, it is the arc of a great circle comprehended between the *positive poles* of rotation.

275. **PROB.** Given $\omega'_1, \omega'_2, \omega'_3$, the angular velocities of a rigid body, estimated respectively about the axes of x', y', z' , which are fixed in space, to find the linear velocities of any particle parallel to the same axes.

Let x', y', z' be the co-ordinates of any particle P referred to the fixed point as origin; r_1 the distance of P from the axis of x' ; θ_1 the inclination of the line r_1 to the plane $x'y'$;

$$\therefore y' = r_1 \cos \theta_1 \text{ and } z' = r_1 \sin \theta_1.$$

Differentiating these equations with regard to t , remembering that $d_t \theta_1 = \omega'_1$, we have

$$\begin{aligned} d_t y' &= -r_1 \sin \theta_1 \cdot \omega'_1 \text{ and } d_t z' = r_1 \cos \theta_1 \cdot \omega'_1 \\ &= -z' \omega'_1 \qquad \qquad \qquad = y' \omega'_1, \end{aligned}$$

that is, the angular velocity ω'_1 produces in P a linear velocity

parallel to $x' = 0$,

parallel to $y' = -z' \omega'_1$,

parallel to $z' = y' \omega'_1$.

Similarly, the angular velocity ω'_2 produces in P a linear velocity

parallel to $x' = z' \omega'_2$,

parallel to $y' = 0$,

parallel to $z' = -x' \omega'_2$:

and, the angular velocity ω'_3 produces in P a linear velocity

parallel to $x' = -y' \omega'_3$

parallel to $y' = x' \omega'_3$.

parallel to $z' = 0$.

Consequently using $d_t x'$, $d_t y'$, $d_t z'$ to denote the *whole* linear velocities of P , and collecting the partial velocities from the equations just given, we obtain

$$d_t x' = z' \omega'_2 - y' \omega'_3,$$

$$d_t y' = x' \omega'_3 - z' \omega'_1,$$

$$d_t z' = y' \omega'_1 - x' \omega'_2.$$

276. PROB. To find the position of the axis of instantaneous rotation, and the angular velocity about it.

By the nature of the case, a rigid body having a fixed point can at the same instant rotate only about one axis passing through that point: the position of this axis may be however perpetually changing both in the body and in space; we have to determine its position in terms of $\omega'_1, \omega'_2, \omega'_3$, supposed known; which we may accomplish by considering that every point in the axis of rotation is linearly at rest; consequently all the points which satisfy the three equations

$$0 = x' \omega'_2 - y' \omega'_3,$$

$$0 = x' \omega'_3 - z' \omega'_1,$$

$$0 = y' \omega'_1 - z' \omega'_2,$$

or the more simple equations

$$\frac{x'}{\omega'_1} = \frac{y'}{\omega'_2} = \frac{z'}{\omega'_3},$$

are in the axis of instantaneous rotation. If therefore $\alpha' \beta' \gamma'$, be the inclinations of this axis to the co-ordinate axes

$$\cos \alpha' = \frac{\omega'_1}{\sqrt{\omega'^2_1 + \omega'^2_2 + \omega'^2_3}},$$

$$\cos \beta' = \frac{\omega'_2}{\sqrt{\omega'^2_1 + \omega'^2_2 + \omega'^2_3}},$$

$$\cos \gamma' = \frac{\omega'_3}{\sqrt{\omega'^2_1 + \omega'^2_2 + \omega'^2_3}},$$

which give the position of it. To find the angular velocity (ω) about it, we observe that, since the body is rigid the angular velocity of every particle is the same; let us therefore, to render the investigation as simple as possible, find the angular velocity of a particle P' which is situated in the axis of x' . Since the co-ordinates of P' are $x', 0$, its linear velocities parallel to the co-ordinate axes are

$$0, \quad x' \omega'_3, \quad -x' \omega'_2$$

and consequently its whole linear velocity

$$= (x'^2 \omega'^2_2 + x'^2 \omega'^2_3)^{\frac{1}{2}} \dots \dots \dots \text{Art. 18.}$$

But the distance of P' from the axis of rotation (which is the radius of the circle which P' describes as the body rotates)

$$\begin{aligned}
 &= x' \sin \alpha' \\
 &= x' \left(\frac{\omega_2'^2 + \omega_3'^2}{\omega_1'^2 + \omega_2'^2 + \omega_3'^2} \right)^{\frac{1}{2}}; \\
 \therefore \omega &= \frac{\text{linear velocity}}{\text{radius}} \\
 &= (\omega_1'^2 + \omega_2'^2 + \omega_3'^2)^{\frac{1}{2}}.
 \end{aligned}$$

277. Since $\cos \alpha' = \frac{\omega_1'}{\omega}$; therefore $\omega_1' = \omega \cos \alpha'$; and because the axis of x' may be taken in any direction whatever, it follows, that angular velocity about one axis may be resolved into angular velocity about any other axis by multiplying it by the cosine of the angle between their positive poles.

278. Suppose the axes of x', y', z' to have been taken in such a position that, at the moment we are considering the motion, they coincide with the principal axes of the body; then if α, β, γ be the inclinations of the instantaneous axis of rotation to the principal axes, the moment of inertia about it

$$= A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma \dots \dots \dots \text{Art. 243.}$$

And consequently the vis viva of the body (which call h^2)

$$\begin{aligned}
 &= (A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma) \cdot \omega^2 \dots \dots \dots \text{Art. 212.} \\
 &= A (\omega \cos \alpha)^2 + B (\omega \cos \beta)^2 + C (\omega \cos \gamma)^2 \\
 &= A \omega_1'^2 + B \omega_2'^2 + C \omega_3'^2 \dots \dots \dots \text{Art. 277,}
 \end{aligned}$$

$\omega_1, \omega_2, \omega_3$ being used exclusively to denote the angular velocities estimated about the principal axes.

Hence if the body be acted on by no forces, $A \omega_1'^2 + B \omega_2'^2 + C \omega_3'^2$ is constant, which agrees with Art. 213.

279. *To investigate differential equations for determining the angular velocities of a body about its principal axes, the body being acted on by given impressed forces.*

Let the co-ordinate axes fixed in space, be taken in such a position, that at the time t when we are considering the motion, they may coincide with the principal axes of the body.

Let $m_1, m_2, m_3, \dots$ be the particles of the body;

$x_1 y_1 z_1, x_2 y_2 z_2, x_3 y_3 z_3, \dots$ their co-ordinates,

$X_1 Y_1 Z_1, X_2 Y_2 Z_2, X_3 Y_3 Z_3, \dots$ the accelerating forces impressed upon them.

Reversing the effective forces, they will balance the impressed forces. Now since the body has a fixed point, the conditions of equilibrium are, that the moments of the effective and impressed forces about the co-ordinate axes must be respectively equal. Hence the three following equations for the motion of the body,

$$\Sigma \{m (y d_t^2 z - z d_t^2 y)\} = \Sigma \{m (Z y - Y z)\} = L \text{ suppose,}$$

$$\Sigma \{m (x d_t^2 z - z d_t^2 x)\} = \Sigma \{m (X z - Z x)\} = M \dots\dots$$

$$\Sigma \{m (x d_t^2 y - y d_t^2 x)\} = \Sigma \{m (Y x - X y)\} = N \dots\dots$$

We must endeavour to simplify these equations by introducing the angular velocities about the co-ordinate axes, for these are common to every particle of the system.

Now by Art. 275,

$$d_t z = y \omega_1 - x \omega_2;$$

$$\therefore d_t^2 z = y d_t \omega_1 + \omega_1 d_t y - x d_t \omega_2 - \omega_2 d_t x$$

$$= y d_t \omega + \omega_1 (x \omega_3 - z \omega_1) - x d_t \omega_2 - \omega_2 (z \omega_3 - y \omega_1)$$

$$\therefore y d_t^2 z = y^2 (d_t \omega_1 + \omega_2 \omega_3) - x y (d_t \omega_2 - \omega_1 \omega_3) - (\omega_1^2 + \omega_2^2) y z;$$

$$\therefore \Sigma (m y d_t^2 z) = \Sigma (m y^2) \cdot (d_t \omega_1 + \omega_2 \omega_3),$$

$$\text{because } \Sigma (m x y) = 0, \text{ and } \Sigma (m y z) = 0.$$

In a similar manner it may be shewn that

$$\Sigma (m z d_t^2 y) = \Sigma (m s^2) \cdot (-d_t \omega_1 + \omega_2 \omega_3);$$

$$\therefore \Sigma \{m (y d_t^2 z - s d_t^2 y)\}$$

$$= \Sigma m (y^2 + s^2) \cdot d_t \omega_1 + \Sigma \{m (y^2 - s^2)\} \cdot \omega_2 \omega_3,$$

$$\text{or, } L = A d_t \omega_1 + (C - B) \omega_2 \omega_3, \dots \dots \text{Art. 243.}$$

Similarly,

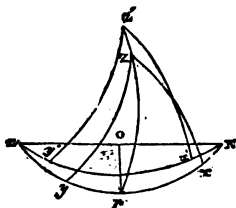
$$M = B d_t \omega_2 + (A - C) \omega_1 \omega_3,$$

$$\text{and } N = C d_t \omega_3 + (B - A) \omega_1 \omega_2.$$

By the integration of these three equations, we should obtain the angular velocities of the body about its principal axes; we shall proceed to shew how when these are known, the position of the body in space may be determined.

280. *To investigate differential equations for determining the position of the body from the angular velocities about its principal axes.*

Let the fixed point, or origin of co-ordinates be the centre of a spherical surface of radius unity. And at the time t let this surface intersect the principal axes in the points x, y, z ; and other co-ordinate axes having the same origin and being fixed in space, in the points x', y', z' . Join these points by arcs of great circles as in the figure; and let Nn be the node line or intersection of the planes $xy, x'y'$.



$$\text{Let } \phi = Nx, \psi = Nx', \theta = xNx'.$$

The point N is the intersection of the arcs $yx, y'x'$ produced; it is not a fixed point in the body and therefore does not necessarily partake of its angular motion. Since it is always in the fixed plane $x'y'$, it has only an angular velocity $d_t \psi$, and this takes place about Os' so as to increase ψ , that is, in a *negative* direction, so that s' is its *negative* pole. Consequently the angular velocity of N about Os

$$\begin{aligned}
 &= d_t \psi \cdot \cos (\pi - \pi \pi') \dots \dots \dots \text{Art. 277} \\
 &= - d_t \psi \cdot \cos \theta.
 \end{aligned}$$

Now $d_t \phi$ = relative angular velocity of x and N about Ox

$$\begin{aligned}
 &= (\text{angular velocity of } x - \text{angular velocity of } N) \\
 &\quad \text{about } Ox \\
 &= \omega_3 + d_t \psi \cdot \cos \theta.
 \end{aligned}$$

Again,

$$\begin{aligned}
 d_t \theta = d_t (\pi \pi') &= \text{angular velocity of the body about } On \\
 &= \omega_1 \cos ny + \omega_2 \cos ny + \omega_3 \cos n\pi \dots \dots \text{Art. 277} \\
 &= - \omega_1 \cos \phi + \omega_2 \sin \phi \dots \dots (1).
 \end{aligned}$$

Again, since N and x are always in a plane passing through Op , their angular velocities about Op are equal.

But the angular velocity of N about Op

$$\begin{aligned}
 &= d_t \psi \cdot \cos (\pi - \pi' p) \dots \dots \dots \text{Art. 277} \\
 &= d_t \psi \cdot \sin \theta.
 \end{aligned}$$

And the angular velocity of x (a point in the body) about Op

$$\begin{aligned}
 &= \omega_1 \cos px + \omega_2 \cos py + \omega_3 \cos p\pi \dots \dots \dots \text{Art. 277} \\
 &= \omega_1 \sin \phi + \omega_2 \cos \phi ; \\
 \therefore d_t \psi \cdot \sin \theta &= \omega_1 \sin \phi + \omega_2 \cos \phi \dots \dots \dots (2).
 \end{aligned}$$

$$\text{Consequently } d_t \phi = \omega_3 + \frac{\cos \theta}{\sin \theta} (\omega_1 \sin \phi + \omega_2 \cos \phi) \dots (3).$$

From the equations marked (1) (2) (3) we are to obtain the values of ϕ , ψ , θ by integration, when the position of the body will be known.

281. PROP. *When the body is acted on by no forces there exists a certain line, determinable from the motion, which is fixed in space.*

For refer the body to three rectangular axes of co-ordinates Ox' , Oy' , Oz' fixed in space. Then the equations of motion are

$$0 = \Sigma \{m (y' d_t^2 x' - x' d_t^2 y')\},$$

$$0 = \Sigma \{m (x' d_t^2 z' - z' d_t^2 x')\},$$

$$0 = \Sigma \{m (x' d_t^2 y' - y' d_t^2 x')\}.$$

By integrating each of these once we obtain

$$h_1 = \Sigma \{m (y' d_t x' - x' d_t y')\},$$

$$h_2 = \Sigma \{m (x' d_t z' - z' d_t x')\},$$

$$h_3 = \Sigma \{m (x' d_t y' - y' d_t x')\}.$$

But $\Sigma \{m (y' d_t x' - x' d_t y')\}$ is equal to the sum of the products of each particle into twice the projection on $y'x'$ of the area described by its radius vector in a unit of time. In order to find this sum we must find the sums for the three principal planes of the body. For the plane yx it is equal to

$$\begin{aligned} & \Sigma \{m (y d_t x - x d_t y)\} \\ &= \Sigma (m r^2 d_t \theta') \\ &= \Sigma (m r^2) \cdot d_t \theta', \\ &= A \omega_1; \end{aligned}$$

r being equal to $(y^2 + x^2)^{\frac{1}{2}}$ and θ' being the inclination of r to the plane xy , so that $d_t \theta' = \omega_1$ and $\Sigma (m r^2) = A$. Similarly, the required sums for the other principal planes are $B\omega_2$ and $C\omega_3$. The projection of these (depending only on the projection of areas) upon the plane $y'x'$ is, by the property proved in *Analytical Geometry of Three Dimensions*, equal to $\Sigma \{m (y' d_t x' - x' d_t y')\}$.

Let therefore a, b, c be the cosines of the inclination of the axis of x' to the axes of x, y, z ; and suppose $a'b'c'$, $a''b''c''$, similar quantities for the axes of y' and z' ;

$$\therefore h_1 = A\omega_1 a + B\omega_2 b + C\omega_3 c.$$

$$\text{Similarly, } h_2 = A\omega_1 a' + B\omega_2 b' + C\omega_3 c', \\ \text{and } h_3 = A\omega_1 a'' + B\omega_2 b'' + C\omega_3 c''.$$

Adding the squares of these quantities together and writing h^2 for $h_1^2 + h_2^2 + h_3^2$, we have

$$h^2 = A^2 \omega_1^2 + B^2 \omega_2^2 + C^2 \omega_3^2.$$

Wherefore a line inclined to the axes of x', y', z' at angles

$$\cos^{-1}\left(\frac{h_1}{h}\right), \quad \cos^{-1}\left(\frac{h_2}{h}\right), \quad \cos^{-1}\left(\frac{h_3}{h}\right)$$

is fixed in position in space during the motion of the body.

282. *PROB. To find the inclination of this line to the principal axes of the body, and the angular velocity about it.*

The cosine of the angle which this line makes with Ox is

$$\frac{h_1}{h} \cdot a + \frac{h_2}{h} \cdot a' + \frac{h_3}{h} \cdot a''$$

which is equal to $\frac{A\omega_1}{h}$ by substituting for $h_1 h_2 h_3$.

Similarly, the cosines of the angles which it makes with Oy and Oz are

$$\frac{B\omega_2}{h} \quad \text{and} \quad \frac{C\omega_3}{h}.$$

Again, the cosines of the inclinations of the axis of instantaneous rotation to Ox, Oy, Oz are

$$\frac{\omega_1}{\omega}, \quad \frac{\omega_2}{\omega}, \quad \frac{\omega_3}{\omega} \dots\dots\dots \text{Art. 277.}$$

Consequently the cosine of the inclination of the proposed line to the axis of instantaneous rotation is

$$\frac{A\omega_1}{h} \cdot \frac{\omega_1}{\omega} + \frac{B\omega_2}{h} \cdot \frac{\omega_2}{\omega} + \frac{C\omega_3}{h} \cdot \frac{\omega_3}{\omega} \\ = \frac{k^2}{h\omega} \dots\dots\dots \text{Art. 278.}$$

And the angular velocity required, being the resolved part of ω , is $\frac{k^2}{h}$ Art. 277, which is constant.

283. Since this line is fixed in space, and its position can be found from the initial circumstances of the motion, we shall be enabled to simplify our results by taking it for the axis of x' . Hence

$$\frac{A\omega_1}{h} = a'' = \cos x'x = -\sin \theta \sin \phi,$$

$$\frac{B\omega_2}{h} = b'' = \cos x'y = -\sin \theta \cos \phi,$$

$$\frac{C\omega_3}{h} = c'' = \cos x'x = \cos \theta.$$

Upon this hypothesis the angular velocity of the body about Ox' is uniform and equal to $\frac{k^2}{h}$. (Art. 282).

284. *PROB. To determine the motion when A, B, C are all equal, and the body is acted on by no forces.*

The equations of Art. 279, are reduced to

$$0 = Ad_t\omega_1, \quad 0 = Ad_t\omega_2, \quad 0 = Ad_t\omega_3.$$

Whence it follows, that $\omega_1, \omega_2, \omega_3$ are constant, as is also ω , being equal to $(\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}}$.

Consequently, the axis of instantaneous rotation is a fixed line in the body, and the angular velocity about it is uniform.

Since this line is a principal axis (Art. 247), take it for the sake of simplicity for that of x ; consequently

$$\omega_1 = 0, \quad \omega_2 = 0, \quad \text{and} \quad \omega_3 = \omega;$$

by which supposition the equations of Art. 280, are reduced to

$$d_t\theta = 0,$$

$$d_t\psi \cdot \sin \theta = 0 \quad \text{or} \quad d_t\psi = 0,$$

$$d\phi = \omega.$$

Hence θ and ψ are constant, from which we infer that π and N are fixed in space: that is, the axis of rotation is fixed in space. The equation $d_t\phi = \text{constant}$, shews that the angular velocity is constant, as shewn before.

285. **PROB.** *To determine the motion when A and B are equal, and the body is acted on by no forces.*

The equations of Art. 279 now become

$$0 = d_t\omega_3,$$

$$0 = Ad_t\omega_2 + (A - C)\omega_1\omega_3,$$

$$0 = Ad_t\omega_1 - (A - C)\omega_2\omega_3.$$

The first shews that ω_3 is constant, hence the velocity about $O\pi$ is uniform; multiply the second by ω_3 , the third by ω_1 , and add;

$$\therefore 0 = A(\omega_1d_t\omega_1 + \omega_2d_t\omega_2);$$

$$\therefore \omega_1^2 + \omega_2^2 = \text{constant};$$

$$\therefore \omega^2 = (\omega_1^2 + \omega_2^2) + \omega_3^2 = \text{constant},$$

$$\text{and } \frac{\omega_3}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}} = \text{constant}.$$

Consequently the body revolves uniformly about the axis of instantaneous rotation; and this line is inclined to $O\pi$ at a constant angle. (Art. 276). The axis of instantaneous rotation is therefore either a fixed line in the body, or it describes about $O\pi$ a conical surface, the cosine of whose semi-vertical angle is $\frac{\omega_3}{\omega}$. We shall shew that the latter is the case.

The last equation of Art. 283 shews that θ or $\pi\pi'$ is constant, and the first two shew that θ is negative and $\sin \theta$ equal to $-\frac{A}{h}(\omega_1^2 + \omega_2^2)^{\frac{1}{2}}$; and

$$\omega_1 = (\omega_1^2 + \omega_2^2)^{\frac{1}{2}} \cdot \sin \phi,$$

$$\text{and } \omega_2 = (\omega_1^2 + \omega_2^2)^{\frac{1}{2}} \cdot \cos \phi;$$

which being written in equation (2) of Art. 280 give

$$d_t \psi \cdot \sin \theta = (\omega_1^2 + \omega_2^2)^{\frac{1}{2}};$$

$$\therefore d_t \psi = -\frac{h}{A}.$$

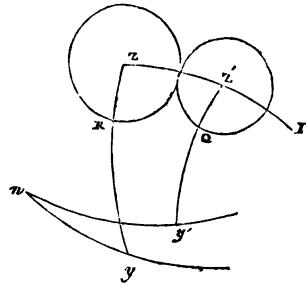
Now $d_t \psi = d_t (y' s' s)$, consequently the axis Ox revolves round Ox' with a uniform angular velocity $\frac{h}{A}$ describing a conical surface whose semi-vertical angle is θ . The linear velocity of $s = d_t \psi \cdot \sin ss' = (\omega_1^2 + \omega_2^2)^{\frac{1}{2}}$.

The motion may be mechanically imitated thus:

With the centres s' and s describe upon the surface of the sphere two circles PQ , PR touching each other at P , and of such spherical radii $s'P$, sP that

$$\sin sP : \sin s'P :: \frac{h}{A} : \omega_3$$

$$:: C : A \cos \theta \dots \text{Art. 281.}$$



Let the former circle be fixed in space and the latter fixed in the body. Then if s be made to revolve about s' with the angular velocity $\frac{h}{A}$ while the circle PR rolls upon PQ the motion of the body will be truly represented.

We may remark that the pole of instantaneous rotation is in the arc ss' . For, let I be the pole. Then

$$\cos ION = \cos IOx \cos NOx + \cos IOy \cos NOy + \cos IOz \cos NOz$$

$$= \frac{\omega_1}{\omega} \cdot \cos \phi + \frac{\omega_2}{\omega} \cdot \cos (90^\circ + \phi) + \frac{\omega_3}{\omega} \cdot \cos 90^\circ$$

$$= \frac{\omega_1 \cos \phi - \omega_2 \sin \phi}{\omega}$$

$$= -\frac{d_t \theta}{\omega} \dots \dots \dots \text{Art. 280, (1)}$$

= 0, because θ is constant ;

$$\therefore ION = 90^\circ,$$

that is, I is in the great circle passing through π and π' .

It was proved in Art. 280, that $\cos \pi' I = \frac{k^2}{h\omega}$,

and in Art. 276, that $\cos \pi I = \frac{\omega_3}{\omega}$;

consequently πI will be less or greater than $\pi' I$ according as

$$\frac{\omega_3}{\omega} \text{ is } > \text{ or } < \frac{k^2}{h\omega},$$

$$\text{or } h^2 \omega_3^2 > \text{ or } < k^4,$$

$$\text{or } (A^2 \omega_1^2 + B^2 \omega_2^2 + C^2 \omega_3^2) \omega_3^2 > \text{ or } < (A \omega_1^2 + B \omega_2^2 + C \omega_3^2)^2,$$

$$\text{or } 1 - \frac{C}{A} > \text{ or } < \frac{1}{2} \left(\frac{\omega}{\omega_3} \right)^2.$$

286. *PROB. To determine the motion when the body is acted on by no forces, and A, B, C are all unequal, and the axis of instantaneous rotation nearly coincides with one of the principal axes of the body.*

Let the principal axis with which the axis of instantaneous rotation nearly coincides be that of π . Consequently $\frac{\omega_3}{\omega}$ the cosine of their inclination is nearly equal to unity ;

$$\therefore 1 - \left(\frac{\omega_3}{\omega} \right)^2 \text{ or } \frac{\omega_1^2 + \omega_2^2}{\omega^2} \text{ is very small :}$$

hence ω_1 and ω_2 are very small ; we shall therefore neglect their product and second powers. By this hypothesis the equations of Art. 279 are reduced to

$$0 = C d_1 \omega_3 \text{ nearly, ... (1),}$$

$$0 = B d_1 \omega_2 + (A - C) \omega_3 \omega_1 \text{ (2),}$$

$$0 = A d_1 \omega_1 + (C - B) \omega_3 \omega \text{ (3).}$$

The first shews that ω_3 is very nearly constant; and therefore θ or $\alpha\alpha'$ is nearly constant (Art. 283), for $\cos \theta = \frac{C\omega_3}{h}$. Also because $h^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2$; therefore $h = C\omega_3$ very nearly, so that $\cos \theta = 1$ very nearly, and θ is very small.

Hence the principal axis Ox nearly coincides with Ox' and maintains nearly a constant inclination to it.

Eliminating ω_2 , by differentiation, from the equations (2) (3), we find

$$d_t^2 \omega_1 + n^2 \omega_1 = 0,$$

writing n^2 for $\left(\frac{C}{B} - 1\right)\left(\frac{C}{A} - 1\right)\omega_3^2$ for brevity;

$$\therefore \omega_1 = K \cos (nt + e);$$

$$\therefore d_t \omega_1 = -nK \cdot \sin (nt + e).$$

Substituting this in equation (3), we obtain

$$\omega_2 = \frac{AnK}{(C - B)\omega_3} \cdot \sin (nt + e).$$

Now from Art. 283 we have

$$\frac{A\omega_1}{B\omega_2} = \tan \phi;$$

$$\therefore \tan \phi = \frac{(C - B)\omega_3}{Bn} \cdot \cot (nt + e)$$

$$= \left(\frac{\frac{C}{B} - 1}{\frac{C}{A} - 1}\right)^{\frac{1}{2}} \cdot \cot (nt + e).$$

Again from Art. 280 we have

$$\begin{aligned} d_t \phi &= \omega_3 + d_t \psi \cdot \cos \theta \\ &= \omega_3 + d_t \psi \text{ nearly;} \end{aligned}$$

$$\therefore \psi = \phi - \omega_3 t + e'$$

which completes the determination of the motion.

287. Remark. When $C - A$ and $C - B$ have the same sign, the above expressions are all possible; that is, when Ox is the axis of *greatest* or *least* moment of inertia, the body will go on perpetually revolving about an axis nearly coinciding with it. If Ox should happen to be the axis of mean moment of inertia $C - A$ and $C - B$ have different signs and the equation for ω_1 is of the form

$$d_t^2 \omega_1 - n^2 \omega_1 = 0,$$

the integral of which involving exponential quantities of the forms e^{nt} and e^{-nt} , ω_1 and ω_2 will perpetually increase with the increase of t , and consequently though the body were originally made to revolve about an axis nearly coinciding with the mean axis, the axis of rotation would recede continually farther from it, and the above investigation would cease to be applicable.

288. If at the commencement of the motion, the body revolve about a principal axis, $\omega_1 = 0$ and $\omega_2 = 0$ when $t = 0$; which can only be satisfied by making $K = 0$, in which case $\omega_1 = 0$ and $\omega_2 = 0$ always: which shews that the body will revolve perpetually about that axis. On this account the principal axes of inertia are sometimes called *natural axes of rotation*.

Conversely if ever $\omega_1 = 0$ and $\omega_2 = 0$ at the *same* instant, the body must of necessity be revolving about a principal axis of inertia: and therefore it always revolved about a principal axis. Wherefore unless a body revolved about a principal axis of inertia at the commencement of the motion it *never* will revolve about one.

289. That the axis of instantaneous rotation may be a fixed line in the body, it is necessary that $\omega_1, \omega_2, \omega_3$ be all constant, in which case

$$d_t \omega_1 = 0, \quad d_t \omega_2 = 0, \quad d_t \omega_3 = 0;$$

$$\therefore 0 = (B - A) \omega_1 \omega_2, \quad 0 = (A - C) \omega_1 \omega_3, \quad 0 = (C - B) \omega_2 \omega_3,$$

to satisfy which it is necessary to suppose two of the three quantities $\omega_1, \omega_2, \omega_3$ evanescent, in which case the body will

revolve about a principal axis. Hence none but a principal axis of inertia can be a perpetual axis of rotation.

290. DEF. On account of the property proved in Art. 287, when the rotation takes place about either of the principal axes of greatest and least moment of inertia it is said to be *stable*; and when it takes place about the mean axis it is said to be *unstable*.

291. We have proved that

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = k^2,$$

$$\text{and } A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = h^2.$$

Eliminating ω_3 we find

$$A(C-A)\omega_1^2 + B(C-B)\omega_2^2 = Ck^2 - h^2.$$

Hence if Ox be an axis of stable rotation, ω_1 and ω_2 cannot increase beyond the limits

$$\left(\frac{Ck^2 - h^2}{AC - A^2}\right)^{\frac{1}{2}} \text{ and } \left(\frac{Ck^2 - h^2}{BC - B^2}\right)^{\frac{1}{2}}$$

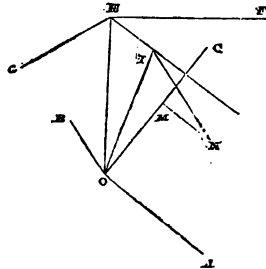
respectively: and as one of them increases the other decreases. Also if they are small at first $Ck^2 - h^2$ is small, and therefore they are *always* small.

292. PROB. To determine the motion generally when the body is acted on by no forces.

Fixed in the body describe an ellipsoid, the semi-axes OA , OB , OC of which coincide with the principal axes of the body: and suppose its equation to be

$$Ax^2 + By^2 + Cz^2 = D^2.$$

At the time t let the axis of instantaneous rotation meet the surface of this ellipsoid in the point I ; and let $OM = x$, $MN = y$, $NI = z$ be the co-ordinates of I .



$$\text{Then } \frac{x}{OI} = \cos IOM = \frac{\omega_3}{\omega} \dots\dots \text{Art. 276.}$$

$$\therefore z = \frac{\omega_3}{\omega} \cdot OI.$$

$$\text{Similarly } y = \frac{\omega_2}{\omega} \cdot OI,$$

$$\text{and } x = \frac{\omega_1}{\omega} \cdot OI;$$

$$\begin{aligned} \therefore D^2 &= (A\omega_1^2 + B\omega_2^2 + C\omega_3^2) \cdot \frac{OI^2}{\omega^2} \\ &= k^2 \cdot \frac{OI^2}{\omega^2} \dots\dots \text{Art. 278;} \end{aligned}$$

$$\therefore OI = \frac{D}{k} \cdot \omega.$$

Let now GEF be a plane touching the ellipsoid in I ; and from O draw OE perpendicular to this tangent plane. Then by Analytical Geometry

$$\begin{aligned} \frac{1}{OE^2} &= \frac{x^2}{OA^4} + \frac{y^2}{OB^4} + \frac{z^2}{OC^4} \\ &= \left(\frac{\omega_1 \cdot OI}{\omega} \right)^2 \cdot \frac{A^2}{D^4} + \left(\frac{\omega_2 \cdot OI}{\omega} \right)^2 \cdot \frac{B^2}{D^4} + \left(\frac{\omega_3 \cdot OI}{\omega} \right)^2 \cdot \frac{C^2}{D^4} \\ &= \frac{A^2 \omega_1^2 + B^2 \omega_2^2 + C^2 \omega_3^2}{k^2 D^2} \\ &= \frac{h^2}{k^2 D^2} \dots\dots \text{Art. 281;} \end{aligned}$$

$$\therefore OE = \frac{kD}{h}.$$

Consequently OE is constant during the whole motion of the body.

Again, by Analytical Geometry, the cosines of the angles at which OE the perpendicular to the tangent plane is inclined to the axes of x, y, z , are

$$\frac{x \cdot OE}{(OA)^2}, \quad \frac{y \cdot OE}{(OB)^2}, \quad \frac{z \cdot OE}{(OC)^2},$$

which are respectively equal to

$$\frac{A\omega_1}{h}, \quad \frac{B\omega_2}{h}, \quad \frac{C\omega_3}{h},$$

which compared with the former part of Art. 282, shew that OE is the line which was there proved to be *fixed* in space, consequently the tangent plane GEF is fixed in space, and as it touches the ellipsoid in the point I , such that OI is the axis of instantaneous rotation, it appears that as the body moves, the ellipsoid rolls upon this fixed plane, in such a manner that the angular velocity of the body about OE is constant (latter part of Art. 282) and equal to $\frac{h^2}{h}$. (See *Camb.*

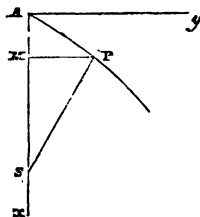
Prob. 1836).

CHAPTER X.

PROBLEMS.

1. *A PARTICLE of matter placed at A is attracted towards the point S by a force which varies as its distance therefrom ; it is also attracted by a constant force acting in parallel lines at right angles to AS. To determine the form of the path which the particle will describe.*

Let μ be the absolute force residing in S ; then if the particle, whose mass denote by m , is at P at the time t from leaving A , it is urged towards S by an accelerating force $= \mu \cdot SP$, and in a direction parallel to Ay by a constant accelerating force, which denote by f . Let AP be the path of m . $x = AM$, $y = MP$, $a = AS$. The force $\mu \cdot SP$ is equivalent to



$$+ \mu \cdot SP \cos ASP = + \mu (a - x) \text{ in the direction } Ax,$$

$$\text{and } - \mu \cdot SP \sin ASP = - \mu y \text{ in the direction } Ay.$$

Consequently, the equations of motion are

$$d_t^2 x = \mu (a - x),$$

$$d_t^2 y = f - \mu y.$$

Multiply these equations by $2 d_t x$, $2 d_t y$ respectively, and integrate ;

$$\therefore (d_t x)^2 = \mu (2 a x - x^2) + C,$$

$$\text{and } (d_t y)^2 = 2 f y - \mu y^2 + C'.$$

At the point A , $x = 0$, $y = 0$, $d_t x = 0$, $d_t y = 0$ (for the body moves from rest), therefore $C = 0$ and $C' = 0$; hence substituting these values of C and C' , and dividing one equation by the other, and extracting the square root

$$\frac{dy}{dx} = \frac{d_y y}{d_x x} = \left(\frac{2 \frac{f}{\mu} y - y^2}{2ax - x^2} \right)^{\frac{1}{2}};$$

$$\therefore \frac{dy}{\left(2 \frac{f}{\mu} y - y^2 \right)^{\frac{1}{2}}} = \frac{dx}{(2ax - x^2)^{\frac{1}{2}}};$$

$$\therefore \text{vers}^{-1} \left(\frac{\mu y}{f} \right) = \text{vers}^{-1} \left(\frac{x}{a} \right) + C.$$

But at A , $x = 0$ and $y = 0$, therefore $C = 0$,
consequently, $\text{vers}^{-1} \left(\frac{\mu y}{f} \right) = \text{vers}^{-1} \left(\frac{x}{a} \right);$

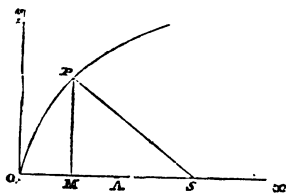
$$\therefore \frac{\mu y}{f} = \frac{x}{a},$$

$$\text{and } y = \frac{f}{\mu a} \cdot x.$$

Hence the path is a straight line commencing at A , and being inclined to the axis of x at an angle equal to $\tan^{-1} \left(\frac{f}{\mu a} \right)$.

2. *A particle of matter is projected from a given point, with a given velocity, in a given direction, and is acted on by a force which varies as the distance, the centre of force also moving uniformly with a given velocity in a direction at right angles to the line of projection. To find the path of the particle.*

Let O the point of projection be the origin of co-ordinates; Oy the direction of projection; Ox the line in which the centre of force S moves; $OM = x$, $MP = y$ the co-ordinates of the particle at the time t , at which time the centre of force is at S : let A be the position of S when the body was projected. $a = OA$: u = the velocity of projection, u' = velocity of S ,



μ = the absolute force. Then P is urged towards S with a force equal to $\mu \cdot SP$, which is equivalent to

$$\mu \cdot SP \cos PSM = \mu \cdot MS = \mu \cdot (a + u't - x),$$

$$\text{and } -\mu \cdot SP \sin PSM = -\mu \cdot PM = -\mu y,$$

in the directions Ox , Oy respectively ;

$$\therefore d_t^2 x = \mu(a + u't - x) \dots (1),$$

$$\text{and } d_t^2 y = -\mu y \dots (2).$$

$$\text{Now, since } d_t(a + u't - x) = u' - d_t x;$$

$$\therefore d_t^2(a + u't - x) = -d_t^2 x,$$

and equation (1) may be written thus,

$$d_t^2(a + u't - x) = -\mu(a + u't - x).$$

This equation is of the same form as (2) and may be integrated by the same method. The student will find little difficulty in proving that

$$x = a + u't - \frac{u'}{\sqrt{\mu}} \sin(t\sqrt{\mu}) - a \cos(t\sqrt{\mu}),$$

$$\text{and } y = \frac{u}{\sqrt{\mu}} \sin(t\sqrt{\mu}).$$

3. *Two weights whose masses are m and m' are suspended from a fixed point by an elastic string; when they are in a state of equilibrium m' suddenly falls off. Determine the time of one oscillation of m .*

Let A be the point from which they were suspended ;
 P the position of m at any time t from the commencement of its motion,

a = the natural (or unstretched) length of the string,

$x = AP$ its length at the time t ,

T = its tension at the same time,

e = the constant which measures the elasticity, which is such that

$$x = (1 + eT)a;$$

$$\therefore T = \frac{x - a}{ae}.$$

This tension is equivalent to an accelerating force $\frac{T}{m}$ urging the body *upwards*; m is also acted on by (g) the force of gravity *downwards*;

$$\therefore d_t^2 x = g - \frac{T}{m} = g - \frac{x - a}{mae};$$

$$\therefore d_t^2 x + \frac{1}{mae} \cdot x = g + \frac{1}{me}.$$

This is an equation of vibratory motion (Art. 152).

The oscillations are therefore isochronous and the time of one oscillation is equal to

$$\pi\sqrt{mae}.$$

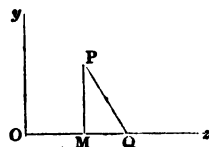
4. A particle P is fastened to one end of a very fine inextensible string PQ (supposed imponderable), and laid upon a perfectly smooth horizontal table yOx : the other end Q is drawn uniformly along the right line Ox . To determine the motion of P .

Let m = the mass of the particle,
 $x = OM$, $y = MP$ the co-ordinates
 of P at the time t ,

u = velocity of Q along Ox ,

$a = PQ$, $\theta = OQP$,

T = the tension of the string.



Then supposing Q to start from O , $OQ = ut$, and

$$x = ut - a \cos \theta, \quad y = a \sin \theta.$$

There is no force upon P besides the tension T , which is equivalent to an accelerating force $\frac{T}{m}$ in the direction PQ . Consequently the equations of P 's motion are

$$d_t^2 x = + \frac{T}{m} \cos \theta,$$

$$d_t^2 y = - \frac{T}{m} \sin \theta.$$

But because $x = ut - a \cos \theta$;

$$\therefore d_t^2 x = a \sin \theta d_t^2 \theta + a \cos \theta (d_t \theta)^2.$$

$$\text{Similarly } d_t^2 y = a \cos \theta d_t^2 \theta - a \sin \theta (d_t \theta)^2.$$

By eliminating T from the equations of motion we obtain

$$\begin{aligned} 0 &= \sin \theta d_t^2 x + \cos \theta d_t^2 y \\ &= a d_t^2 \theta, \text{ by substitution;} \\ \therefore d_t \theta &= \text{constant.} \end{aligned}$$

It appears therefore that as Q moves uniformly along Ox , P revolves uniformly about Q ; hence the locus of P is a trochoid or cycloid, the nature of which is, that the centre (Q) of its generating circle moves uniformly along a right line (Ox) while the generating point (P) revolves uniformly about the centre.

If we wish to determine the values of x and y in terms of t we may proceed thus. Since $d_t \theta$ is constant, represent it by ω ;

$$\therefore \theta = \omega t + C, \text{ by integration.}$$

Now when Q is at O , suppose that P is in the line Oy ; then $\theta = \frac{\pi}{2}$ when $t = 0$, therefore $C = \frac{\pi}{2}$;

$$\therefore \theta = \omega t + \frac{\pi}{2};$$

$$\therefore x = ut + a \sin \omega t \text{ and } y = a \cos \omega t.$$

We have yet to find the value of ω . By differentiation we obtain

$$d_t x = u + a \omega \cos \omega t.$$

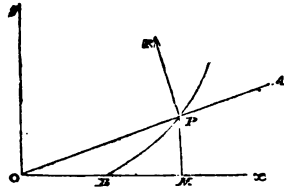
But $d_t x = 0$ when $t = 0$, and therefore $0 = u + a \omega$;

consequently $\omega = -\frac{u}{a}$;

$$\left. \begin{aligned} \therefore x &= ut - a \sin \frac{ut}{a}, \\ \text{and } y &= a \cos \frac{ut}{a}. \end{aligned} \right\}$$

5. *A very small ring being placed upon a smooth straight rod, the rod is made to revolve uniformly (about one end) in a horizontal plane. To determine the motion of the ring.*

Let Ox and B be the positions of the rod and ring at the commencement of the motion; OA and P their positions at t . Put $r = OP$, $a = OB$, $\omega =$ the angular velocity of the rod. The ring P is acted on only by R the pressure of the rod against it in the direction PR at right angles to OA : this may be avoided by estimating the forces in the direction of the rod. By Art. 93, as there is no impressed force on the ring in the direction of the rod, we have



$$d_t^2 r - \omega^2 r = 0, \dots\dots (1),$$

which may be written,

$$d_t(d_t r - \omega r) + \omega(d_t r - \omega r) = 0.$$

Multiply this by $e^{\omega t}$ and integrate, then

$$e^{\omega t}(d_t r - \omega r) = C \dots\dots (2).$$

Now as equation (1) is not altered by writing $-\omega$ for ω , we may obtain another integral by writing $-\omega$ for ω in equation (2);

$$\therefore e^{-\omega t}(d_t r + \omega r) = C'.$$

These integrals are to be corrected by making $r = a$ and $d_t r = 0$, when $t = 0$; and

$$\therefore C = -\omega a, \text{ and } C' = \omega a;$$

$$\text{consequently } d_t r - \omega r = -\omega a e^{-\omega t},$$

$$\text{and } d_t r + \omega r = \omega a e^{\omega t};$$

and therefore by subtraction, and dividing by 2ω ,

$$r = \frac{a}{2} (e^{\omega t} + e^{-\omega t}).$$

6. *The rod in last problem being supposed to move in a vertical plane yOx , to determine the motion.*

The equation of motion is (by Art. 93)

$$d_t^2 r - \omega^2 r = -g \sin \omega t;$$

$$\text{and } \therefore r = \frac{a}{2} (e^{\omega t} + e^{-\omega t}) - \frac{g}{4\omega^2} (e^{\omega t} - e^{-\omega t} - 2 \sin \omega t).$$

7. *A particle descends along the convex side of a smooth parabolic arc, the axis of which is vertical, having been projected horizontally at the vertex with a given velocity. To find the pressure on the curve.*

Let Ax be the vertical axis of the parabola; P the position of the particle at any time t . $x = AM$, $y = MP$, $4a =$ the latus rectum of the parabola: $m =$ the mass of the particle, $u =$ the given horizontal velocity at A . Then

$$y^2 = 4ax.$$

$$\begin{aligned} \text{In general, } (\text{vel.})^2 &= 2 \int (X dx + Y dy) \\ &= 2 \int (g dx), \text{ in this case;} \\ &= 2gx + C. \end{aligned}$$

Now at A , $(\text{vel.}) = u$, $x = 0$, and therefore $C = u^2$;

$$\therefore (\text{vel.})^2 = 2gx + u^2.$$

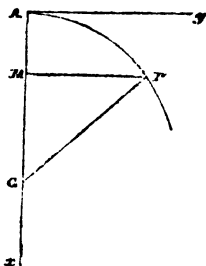
The radius of curvature at P

$$= \frac{PG^2}{(2a)^2} = 2 \cdot \frac{(a+x)^{\frac{3}{2}}}{\sqrt{a}}.$$

Consequently, the diminution of pressure due to centrifugal force

$$= \frac{m \cdot (\text{vel.})^2}{\text{rad. curv.}} = \frac{m}{2} \cdot \frac{(2gx + u^2)\sqrt{a}}{(a+x)^{\frac{3}{2}}}.$$

But the pressure due to the impressed force of gravity



$$= mg \cdot \cos PGA = mg \left(\frac{a}{a+x} \right)^{\frac{1}{2}},$$

$$\therefore \tan PGA = \frac{MP}{MG} = \frac{y}{2a} = \left(\frac{x}{a} \right)^{\frac{1}{2}}.$$

Consequently, the whole pressure at P , (Art. 147)

$$= mg \left(\frac{a}{a+x} \right)^{\frac{1}{2}} - \frac{m}{2} \cdot \frac{(2gx + u^2) \sqrt{a}}{(a+x)^{\frac{1}{2}}}$$

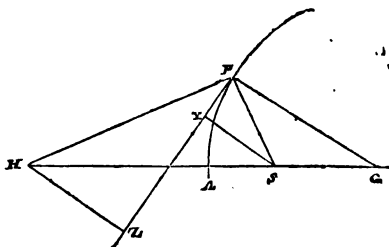
$$= \frac{m}{2} \cdot \frac{(2ag - u^2) \sqrt{a}}{(a+x)^{\frac{1}{2}}}.$$

Remark. If the velocity of projection be such that $u^2 = 2ag$, there will be no pressure on any part of the curve.

8. *A particle attracted to two centres of force varying inversely as the square of the distance, will oscillate in the arc of an hyperbola of which they are the foci, supposing it to have been originally at rest in such a position as to be attracted equally by each.*

Let S, H be the foci of the hyperbola AP ; P the place of the body at any time; PG a normal, and PZ a tangent at P ; SY, HZ perpendiculars upon PZ .

$r = SP$, $r' = HP$; R, R' the original values of SP and HP ; then if μ, μ' be the absolute forces at S and H respectively,



$$\frac{\mu}{R^2} = \frac{\mu'}{R'^2}, \text{ by the question.}$$

Let a, b be the semiaxes of the hyperbola, and e its eccentricity: m = the mass of the particle, v = its velocity at P , s = the arc described from the position of rest.

Suppose the particle to move on the convex side of the

smooth curve PA , and let us find the pressure upon the curve upon this supposition.

To find the velocity v .

$$\text{From Art. 147, } v^2 = 2 \int (Xdx + Ydy).$$

But it appears from Art. 144, that the value of $\int (Xdx + Ydy)$ is the same as if the particle had fallen freely through the space $R - r$ directly towards S , and the space $R' - r'$ directly towards H ;

$$\therefore v^2 = 2 \int \left(-\frac{\mu dr}{r^2} \right) + 2 \int \left(-\frac{\mu' dr'}{r'^2} \right).$$

The former integral is to be taken between the limits $r = R$, and $r = r$; and the latter between $r' = R'$ and $r' = r'$;

$$\therefore v^2 = 2\mu \left(\frac{1}{r} - \frac{1}{R} \right) + 2\mu' \left(\frac{1}{r'} - \frac{1}{R'} \right).$$

The radius of curvature at P

$$= \frac{(e^2 x^2 - a^2)^{\frac{1}{2}}}{ab} = \frac{(rr')^{\frac{1}{2}}}{ab}.$$

Consequently, the diminution of pressure due to centrifugal force

$$= m \cdot \frac{(\text{vel})^2}{\text{rad. curv.}} = \frac{2mab}{(rr')^{\frac{1}{2}}} \cdot \left\{ \mu \left(\frac{1}{r} - \frac{1}{R} \right) + \mu' \left(\frac{1}{r'} - \frac{1}{R'} \right) \right\}.$$

Also the pressure due to the forces at S and H

$$= \frac{\mu m}{r^2} \cdot \sin SPY - \frac{\mu' m}{r'^2} \cdot \sin HPZ$$

$$= \frac{\mu m}{r^2} \cdot SY - \frac{\mu' m}{r'^2} \cdot HZ$$

$$= \frac{\mu m}{r^2} \cdot b \left(\frac{r}{r'} \right)^{\frac{1}{2}} - \frac{\mu' m}{r'^2} \cdot b \left(\frac{r'}{r} \right)^{\frac{1}{2}} \dots \text{by Conic Sections,}$$

$$= \frac{mb}{(rr')^{\frac{1}{2}}} \cdot \left(\frac{\mu r'}{r} - \frac{\mu' r}{r'} \right).$$

Consequently, the whole pressure upon the curve

$$\begin{aligned}
 &= \frac{mb}{(rr')^{\frac{1}{2}}} \cdot \left\{ \frac{\mu r'}{r} - \frac{\mu' r}{r'} - 2a\mu \left(\frac{1}{r} - \frac{1}{R} \right) - 2a\mu' \left(\frac{1}{r'} - \frac{1}{R'} \right) \right\} \\
 &= \frac{mb}{(rr')^{\frac{1}{2}}} \cdot \left\{ \frac{\mu}{r} (r' - 2a) - \frac{\mu'}{r'} (r + 2a) + 2a \left(\frac{\mu}{R} + \frac{\mu'}{R'} \right) \right\} \\
 &= \frac{mb}{(rr')^{\frac{1}{2}}} \cdot \left\{ \mu - \mu' + \mu \cdot \frac{2a}{R} + \mu' \cdot \frac{2a}{R'} \right\}; \because r' - r = 2a, \\
 &= \frac{mb}{(rr')^{\frac{1}{2}}} \cdot \left\{ \mu \frac{R'}{R} - \mu' \frac{R}{R'} \right\}; \because R' - R = 2a, \\
 &= \frac{mbRR'}{(rr')^{\frac{1}{2}}} \cdot \left(\frac{\mu}{R^2} - \frac{\mu'}{R'^2} \right) = 0.
 \end{aligned}$$

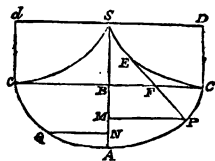
Since there is no pressure on the curve, the curve might be removed without affecting the motion; and, therefore, the body will oscillate in the hyperbolic arc as stated in the question.

9. *A particle oscillates in the whole cycloid CAc; shew that the point F, which is the intersection of the string EP with the base Cc, moves uniformly from C to c.*

Let $x = AM$, $y = MP$, $s = AP$, $2a = AB$. Then the equation of the normal PE is

$$y - y' = -\frac{1}{d_x y} (x - x').$$

But at the point F , $x' = AB = 2a$,
and $y' = BF$;



$$\therefore BF = y - \frac{2a - x}{d_x y}.$$

$$\text{Now, } y = \sqrt{2ax - x^2} + a \text{ vers}^{-1} \frac{x}{a};$$

$$\therefore d_x y = \left(\frac{2a - x}{x} \right)^{\frac{1}{2}}, \text{ by differentiation;}$$

$$\therefore BF = y - \sqrt{2ax - x^2}$$

$$= a \operatorname{vers}^{-1} \frac{x}{a};$$

$$\text{and } \therefore CF = BC - BF$$

$$= a\pi - a \operatorname{vers}^{-1} \frac{x}{a}.$$

Consequently, the velocity of the point F .

$$= d_t(CF)$$

$$= - \frac{a d_t x}{\sqrt{2ax - x^2}}$$

$$= - \frac{a d_t x d_t s}{\sqrt{2ax - x^2}}.$$

$$\text{But } (d_t s)^2 = 1 + (d_t y)^2 = \frac{2a}{x},$$

$$\text{and } -d_t s = \text{vel. of } P$$

$$= \sqrt{2g(AB - AM)} \dots \dots \text{Art. 69.}$$

$$= \sqrt{2g(2a - x)}.$$

Hence the velocity of the point F

$$= \frac{a}{\sqrt{2ax - x^2}} \cdot \left(\frac{x}{2a}\right)^{\frac{1}{2}} \cdot \sqrt{2g(2a - x)}$$

$$= \sqrt{ag}, \text{ which is constant.}$$

10. *Two equal bodies which attract each other with forces varying as the square of their distance inversely, are constrained to move in two straight lines at right angles to one another; shew that they will arrive together at the point of intersection of the lines. And having given their distance at the beginning of the motion, find the time to the point of intersection.*

Let OA , OB be the lines; A , B the places of the particles (whose masses are m , m') at the beginning of the motion; P , Q their places after the time t . $x = OP$, $y = OQ$, $r = PQ$, $a = AB$, $\alpha = BAO$. Then (Art. 29) the mutual attraction of m and m' is equivalent to an equal pressure applied to them in opposite directions; i. e. to P in the direction PQ , and to Q in the direction QP . Since each particle of m attracts each particle of m' , the product mm' will denote this pressure when m and m' are at the unit of distance from each other, and then by the question, the mutual pressure at the time t , when their distance is PQ ,

$$= \frac{mm'}{PQ^2} = \frac{mm'}{r^2}.$$

This is equivalent to an accelerating force on m in the direction PQ , equal to $\frac{m'}{r^2}$. (Art. 23). Which being resolved gives $\frac{m'}{r} \cdot \cos QPO = \frac{m'}{r^2} \cdot \frac{x}{r}$ in the direction PO . There being no other force upon m in this direction, the equation for m 's motion is

$$d_t^2 x = - \frac{m'x}{r^3}.$$

Similarly, the equation for m' 's motion is

$$d_t^2 y = - \frac{my}{r^3}.$$

Multiply the former by y and the latter by x , and subtract (observing that $m = m'$);

$$\therefore x d_t^2 y - y d_t^2 x = 0;$$

$$\therefore x d_t y - y d_t x = C, \text{ by integration.}$$

But at the beginning of the motion $d_t x = 0$, $d_t y = 0$, and therefore $C = 0$;

$$\therefore x d_t y - y d_t x = 0.$$

Dividing this equation by x^2 and integrating, we obtain

$$\frac{y}{x} = C.$$

Consequently when $x = 0$, $y = 0$, so that the particles arrive at O together.

Again, since $\frac{y}{x} = \tan QPO$ is constant, QP is parallel to BA , and $QPO = \alpha$;

$$\therefore x = r \cos \alpha, \quad y = r \sin \alpha.$$

By which the equations of motion are reducible to

$$d_t^2 r = -\frac{m}{r^2}.$$

This equation must be integrated from $r = a$ to $r = 0$. Multiply by $2d_t r$ and integrate;

$$\therefore (d_t r)^2 = 2m \left(\frac{1}{r} + C \right).$$

But when $r = a$, $d_t r = 0$, and therefore $C = -\frac{1}{a}$;

$$\therefore d_t r = -\sqrt{2m} \cdot \left(\frac{1}{r} - \frac{1}{a} \right)^{\frac{1}{2}}.$$

The rest of the operation is the same as in Art. 88, in which μ stands for m . Hence, the time required is

$$\frac{a^{\frac{3}{2}} \pi}{2\sqrt{2m}}.$$

11. *Two particles, subject to no external influence, attract each other with a force which varies inversely as the square of their distance, and are projected in any directions: shew how to determine their motions.*

It has been shewn in Art. 184 that the mutual attraction of bodies has no effect upon the motion of their common centre of gravity, which will therefore move uniformly in a straight line. The position of this line, and the velocity

of the centre of gravity in it, may be found by Art. 50 from the known velocities and directions of projection. In Art. 193 it has been proved that the motion relative to the centre of gravity is the same as if the centre of gravity were a fixed point. Hence we must apply to each body a velocity equal and opposite to that of the common centre of gravity, which will reduce this point to rest. Let now m, m' be the masses of the particles; r, r' their respective distances from the centre of gravity at any time t ; mm' the pressure at the unit of distance from each other. Then the accelerating force by which m is urged towards the centre of gravity is

$$\frac{m'}{(r+r')^2}, \text{ which } = \frac{m'^3}{(m+m')^2 r^2}, \text{ because } mr = m'r'.$$

Hence m is urged towards that point by a force which varies inversely as the square of its distance therefrom. Its orbit will therefore be a Conic Section, the focus of which is in the centre of gravity of the system. The same may be said of m' . The exact magnitudes and positions of these orbits may be found as in Art. 126, observing that, since we are seeking orbits *relative* to the centre of gravity, we must use the *relative velocities* of the bodies with respect to that point instead of the absolute velocities in those articles. We may remark that the orbits described about the centre of gravity are necessarily similar, because r has a constant ratio to r' . And by the nature of the case the periodic times are equal.

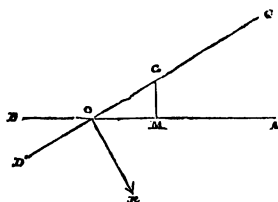
12. *A straight rod moves on a smooth horizontal plane, subject to the condition of always passing through a fixed ring; determine its position.*

Let AB be the original position of the rod, CD its position and G its centre of gravity after the time t , O the fixed point,

$x = OM, y = MG, r = OG, \theta = AOC$.

M = the mass of the rod, Mk^2 = its moment of inertia about G .

The only force which acts upon the rod is the reaction (R) of the point O , which must take



place in the direction OR at right angles to CD , the rod being supposed smooth.

To determine the motion of the point G we may suppose the whole mass collected at G and there acted upon by the force R , (Art. 196) consequently the equations of motion for G are

$$d_t^2 x = \frac{R \cos ROA}{M} = \frac{R}{M} \cdot \frac{y}{r} \dots \dots \dots (1),$$

$$d_t^2 y = - \frac{R \sin ROA}{M} = - \frac{R}{M} \cdot \frac{x}{r} \dots \dots \dots (2).$$

Again, to determine the angular motion about G , we may suppose G fixed, the force R still acting at O to turn the body round (Art. 197);

$$\therefore d_t^2 \theta = \frac{Rr}{Mk^2} \dots \dots \text{Art. 252} \dots \dots \dots (3).$$

Multiply (1) by y and (2) by x and subtract;

$$\begin{aligned} \therefore y d_t^2 x - x d_t^2 y &= \frac{Rr}{M} \\ &= k^2 d_t^2 \theta, \text{ from (3).} \end{aligned}$$

Integrating this equation

$$y d_t x - x d_t y = k^2 d_t \theta + \text{const.}$$

$$\text{or} \quad -r^2 d_t \theta = k^2 d_t \theta + \text{const.}$$

$$\therefore (k^2 + r^2) d_t \theta = C \dots \dots \dots (4).$$

Again, multiply (1) by $2d_t x$, (2) by $2d_t y$, (3) by $2k^2 d_t \theta$, and add, observing that

$$y d_t x - x d_t y + r^2 d_t \theta = 0,$$

$$\text{because } x = r \cos \theta \text{ and } y = r \sin \theta;$$

by this means we obtain

$$2d_t x d_t^2 x + 2d_t y d_t^2 y + 2k^2 d_t \theta d_t^2 \theta = 0,$$

and integrating this,

$$(d_t x)^2 + (d_t y)^2 + k^2 (d_t \theta)^2 = C' \dots \dots (5),$$

$$\text{or } (d_t r)^2 + r^2 (d_t \theta)^2 + k^2 (d_t \theta)^2 = C'.$$

From this and (4) we find

$$d_t t = \left(\frac{k^2 + r^2}{C' k^2 - C^2 + C' r^2} \right)^{\frac{1}{2}}.$$

By the integration of this equation the relation between r and t will be known.

The pressure may be found from the equations (3) and (4); for

$$\frac{Rr}{Mk^2} = d_t^2 \theta = - \frac{2Cr d_t r}{(k^2 + r^2)^2};$$

$$\therefore R = - \frac{2MCK^2}{(k^2 + r^2)^2} \cdot \left(C' - \frac{C^2}{k^2 + r^2} \right)^{\frac{1}{2}}.$$

The negative sign shews that R acts in a direction contrary to that which we have supposed in the figure.

Remark. The preceding, which is somewhat tedious, is the method of solving the question, (so far as is possible in finite terms) from the *equations of motion*. The student will observe however that equations (4) and (5), which are the limits to which we have been able to advance in the solution, are deducible at once from the two principles of the *Conservation of Areas*, and the *Conservation of Vis Viva*.

For, since the only force acting on the system is the force R which passes through the *fixed* point O , the Conservation of Areas holds good. (Art. 191, 3).

Now if the particles of the rod be $m_1, m_2, m_3, \dots$ and their distances from O be $r_1', r_2', r_3', \dots$ then, since

$$r_1'^2 d_t \theta = x_1' d_t y_1' - y_1' d_t x_1',$$

$$r_2'^2 d_t \theta = x_2' d_t y_2' - y_2' d_t x_2',$$

$$\&c. = \&c.$$

$$m_1 r_1'^2 d_i \theta + m_2 r_2'^2 d_i \theta + m_3 r_3'^2 d_i \theta + \dots = h \dots \text{Art. 185.}$$

$$\text{or } \Sigma (m r'^2) \cdot d_i \theta = h.$$

But $\Sigma (m r'^2)$ = the moment of inertia about O .

$$= M k^2 + M r^2 \dots \text{Art. 224;}$$

$$\therefore M (k^2 + r^2) d_i \theta = h,$$

which is equivalent to equation (4).

Again, since the only force acting upon the system is the reaction R of a fixed point, the principle of Vis Viva applies. (Art. 203). And by Art. 213 the Vis Viva is constant. Now the Vis Viva, by Arts. 211 and 212,

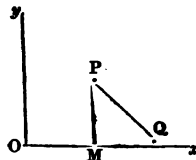
$$= M \{ (d_i x)^2 + (d_i y)^2 \} + M k^2 (d_i \theta)^2;$$

$$\therefore (d_i x)^2 + (d_i y)^2 + k^2 (d_i \theta)^2 = \text{constant},$$

which is equation (5).

13. P and Q are two particles connected by an inflexible line PQ ; Q moves along a straight groove, and PQ on a smooth horizontal plane. Having given the initial position of the rod and the quantity of motion communicated to Q , find the angular velocity of the rod when it coincides with the groove.

Let m, m' be the masses of P and Q , MV the momentum or blow communicated to Q , which to be effective must be in the direction of the groove Ox .



$a = PQ$, $\theta = PQO$, $x' = OQ$, $x = OM$, $y = MP$, T = tension of the rod. The tension produces an accelerating force $\frac{T}{m'}$, in the direction QP , upon Q ; which gives $-\frac{T}{m'} \cos \theta$ in the direction Ox . This being the only force upon Q in that direction, we have for Q 's motion the equation

$$d_i^2 x' = -\frac{T}{m'} \cos \theta.$$

The only force upon P is the accelerating force $\frac{T}{m}$ in the

direction PQ , and therefore the equations of motion for P are

$$d_t^2 x = \frac{T}{m} \cos \theta,$$

$$d_t^2 y = -\frac{T}{m} \sin \theta.$$

These are the only equations of motion which are necessary; there are besides them the geometrical equations of the system, viz.:

$$x' = x + a \cos \theta \text{ and } y = a \sin \theta.$$

Instead however of proceeding further in the solution of the question by the equations of motion, it will perhaps be a more useful example if we employ the general Dynamical principles.

Since there is no force upon the system in the direction Ox , the principle of the conservation of the motion of the centre of gravity holds good in that direction;

$$\therefore (m + m') d_t \bar{x} = MV;$$

$$\therefore m d_t x + m' d_t x' = MV \dots (1),$$

$$\text{because } (m + m') \bar{x} = mx + m' x'.$$

And since the system is acted on by no force besides the reaction of the groove at Q , the Vis Viva is constant. (Art. 213);

$$\therefore m' (d_t x')^2 + m \{ (d_t x)^2 + (d_t y)^2 \} = \text{constant} \dots (2).$$

$$\text{Now } MV = m d_t x + m' d_t (x + a \cos \theta) \dots \text{from (1);}$$

$$\therefore d_t x = \frac{MV + m' a \sin \theta d_t \theta}{m + m'},$$

$$d_t y = a \cos \theta \cdot d_t \theta,$$

$$\text{and } d_t x' = d_t x - a \sin \theta \cdot d_t \theta$$

$$= \frac{MV - m a \sin \theta d_t \theta}{m + m'}.$$

By substituting these values in equation (2) we obtain

$$M^2 V^2 + m a^2 (m' + m \cos^2 \theta) (d_t \theta)^2 = C.$$

To find the value of this constant we must consider the initial state of the motion.

Now since the blow is communicated to Q , we may in finding the angular velocity, consider P fixed for an instant, and the system otherwise free. On this supposition, the moment of the blow about P is $MV \cdot a \sin \alpha$, α being the value of θ when the system is struck; and the moment of inertia of the system about P is $m'a^2$, therefore (Art. 253) the initial angular velocity

$$= \frac{MV \cdot a \sin \alpha}{m'a^2} = \frac{MV \sin \alpha}{m'a}.$$

Hence at the beginning of the motion

$$M^2 V^2 + m a^2 (m' + m \cos^2 \alpha) \cdot \frac{M^2 V^2 \sin^2 \alpha}{m'^2 a^2} = C;$$

$$\therefore m a^2 (m' + m \cos^2 \theta) (d_t \theta)^2 = m a^2 (m' + m \cos^2 \alpha) \cdot \frac{M^2 V^2 \sin^2 \alpha}{m'^2 a^2};$$

$$\therefore d_t \theta = - \frac{MV \sin \alpha}{m'a} \cdot \left(\frac{m' + m \cos^2 \alpha}{m' + m \cos^2 \theta} \right)^{\frac{1}{2}}.$$

When the rod coincides with the groove $\cos^2 \theta = 1$, and consequently the angular velocity required

$$= - \frac{MV \sin \alpha}{m'a} \cdot \left(\frac{m' + m \cos^2 \alpha}{m' + m} \right)^{\frac{1}{2}}.$$

14. *Two weights P, Q are connected by a string (without weight) passing over a pulley; P descending draws Q along a smooth horizontal plane. To determine the pressure of Q upon the plane.*

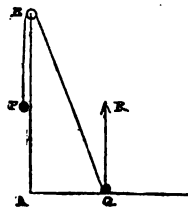
Let m, m' be the masses of P and Q ,

a = the length of the string,

$b = AB, \theta = QBA$,

$x = BP, y = AQ$,

R = the pressure of Q upon the plane.



Then since there are no forces upon the system except gravity which acts upon P , and the tension of the string and reaction at Q , the equation of Vis Viva holds good.

$$\begin{aligned}\text{Now } \Sigma (mv^2) &= m (d_t x)^2 + m' (d_t y)^2, \\ \text{and } 2 \Sigma \{m \int (X dx + Y dy + Z dz)\} &= 2 \int (mg dx) \\ &= 2mgx + C; \\ \therefore m (d_t x)^2 + m' (d_t y)^2 &= 2mgx + C.\end{aligned}$$

But at the beginning of the motion P is at B , and therefore $x = 0$, $d_t x = 0$, $d_t y = 0$, and $C = 0$;

$$\therefore m (d_t x)^2 + m' (d_t y)^2 = 2mgx \dots\dots\dots (1).$$

But from the nature of the machine

$$\begin{aligned}a - x &= BQ = \sqrt{b^2 + y^2}; \\ \therefore d_t x &= \frac{-y d_t y}{\sqrt{b^2 + y^2}}.\end{aligned}$$

This being substituted in equation (1) gives

$$(d_t y)^2 = \frac{2g(a - \sqrt{b^2 + y^2})}{\frac{m'}{m} + \frac{y^2}{b^2 + y^2}} \dots\dots\dots (2).$$

Now since Q has no vertical motion, the forces which act upon it in a vertical direction are such as exactly counteract each other. Hence resolving the tension we have

$$\begin{aligned}0 &= m'g - R - T \cos BQR; \\ \therefore R &= m'g - T \cos BQR.\end{aligned}$$

To find T we must form the equation of motion for Q , which is

$$\begin{aligned}d_t^2 y &= - \frac{T \sin QBA}{m'}; \\ \therefore R &= m'g + m' d_t^2 y \cdot \cot BQR \\ &= m'g + \frac{m'b}{y} \cdot d_t^2 y.\end{aligned}$$

Differentiating equation (2) with regard to t , we obtain the value of $d_t^2 y$, which substituted in the last equation, the result expressed in terms of θ is

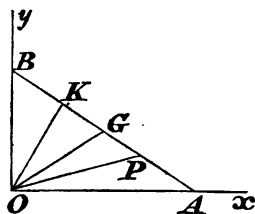
$$R = m'g - m'g \cdot \frac{\cos \theta}{\left(\frac{m'}{m} + \sin^2 \theta\right)} \cdot \left(\frac{m'}{m} + \sin^2 \theta - 2 \cos^2 \theta + \frac{2a}{b} \cos^3 \theta\right).$$

15. A uniform slender rod acted upon by gravity g is placed between two planes (one horizontal and the other vertical) having at their common intersection an attractive force $\propto (\text{dist.})^{-2}$ which at the centre of gravity of the rod always $= \frac{1}{2}g$. If the rod be originally placed in any given position, determine the angular motion.

Let AB be the position of the rod at the time t from the beginning of the motion;

$a = AG = BG = OG$; $\theta = OAB$, M = the mass of the rod, and m = the mass of a small element at P , x, y the co-ordinates of P , $OP = r$. Then since there is no friction the equation of Vis Viva applies;

$$\therefore \Sigma (mv^2) = 2 \Sigma \{m \int (Xdx + Ydy + Zdz)\}.$$



But by Art. 211,

$$\Sigma (mv^2) = \Sigma [m \{(\dot{x})^2 + (\dot{y})^2\}] + \Sigma (m \cdot GP^2) \cdot (\dot{\theta})^2.$$

By the geometrical property of the system

$$\bar{x} = a \cos \theta, \text{ and } \bar{y} = a \sin \theta.$$

And $\Sigma (m \cdot GP^2)$ = moment of inertia of GA and GB about G

$$\begin{aligned} &= \frac{M}{2} \cdot \frac{a^2}{3} + \frac{M}{2} \cdot \frac{a^2}{3} \dots\dots\dots \text{Art. 222,} \\ &= M \cdot \frac{a^2}{3}; \end{aligned}$$

$$\begin{aligned} \therefore \Sigma (mv^2) &= \Sigma \{m a^2 (\dot{\theta})^2\} + M \frac{a^2}{3} \cdot (\dot{\theta})^2 \\ &= \frac{4}{3} M a^2 (\dot{\theta})^2. \end{aligned}$$

Again, the attractive force of O upon P

$$= \frac{g}{2} \cdot \frac{a^2}{OP^2} \dots \text{by the question.}$$

Which is equivalent to

$$-\frac{g}{2} \cdot \frac{a^2}{OP^2} \cdot \frac{x}{OP} \text{ parallel to } Ox,$$

$$\text{and } -\frac{g}{2} \cdot \frac{a^2}{OP^2} \cdot \frac{y}{OP} \text{ parallel to } Oy;$$

$$\therefore X = -\frac{g}{2} \cdot \frac{a^2 x}{OP^3},$$

$$\text{and } Y = -g - \frac{g}{2} \cdot \frac{a^2 y}{OP^3};$$

$$\therefore m(Xdx + Ydy + Zdz) = -\frac{mg}{2} \left\{ \frac{a^2 x dx + a^2 y dy}{OP^3} + 2dy \right\}$$

$$= -\frac{mg}{2} \left\{ \frac{a^2 r dr}{r^3} + 2dy \right\};$$

$$\therefore m \int (Xdx + Ydy + Zdz) = C - \frac{mg}{2} \left\{ -\frac{a^2}{r} + 2y \right\}$$

$$= C - mgy + \frac{mga^2}{2r};$$

$$\therefore 2 \Sigma \{ m \int (Xdx + Ydy + Zdz) \} = C' - 2g \Sigma (my)$$

$$+ ga^2 \Sigma \left(\frac{m}{r} \right).$$

Now $\Sigma(my) = \Sigma m \cdot \bar{y} = Ma \sin \theta$; and to find the value of $\Sigma \left(\frac{m}{r} \right)$ we must proceed as follows. Let fall OK perpendicular to AB ; then $OK = 2a \cos \theta \sin \theta$, and

$$KP = (r^2 - 4a^2 \cos^2 \theta \sin^2 \theta)^{\frac{1}{2}};$$

$\therefore$ an element of the rod at $P = \delta \cdot KP$,

$$\text{and its mass} = M \cdot \frac{\delta \cdot KP}{2a};$$

$$\begin{aligned}
 \therefore \Sigma \left(\frac{m}{r} \right) &= \frac{M}{2a} \cdot \Sigma \left(\frac{\delta \cdot KP}{r} \right) \\
 &= \frac{M}{2a} \cdot \Sigma \frac{\delta r}{(r^2 - 4a^2 \cos^2 \theta \sin^2 \theta)^{\frac{1}{2}}} \\
 &= \frac{M}{2a} \cdot \log_e (r + \sqrt{r^2 - 4a^2 \cos^2 \theta \sin^2 \theta}) + C.
 \end{aligned}$$

If this integral be taken from $r = OK$, to $r = OA$, it will give us the value of $\Sigma \left(\frac{m}{r} \right)$ for the part KA of the rod

$$\begin{aligned}
 &= \frac{M}{2a} \cdot \log_e \frac{1 + \cos \theta}{\sin \theta} \\
 &= \frac{M}{2a} \log_e \cot \frac{\theta}{2}.
 \end{aligned}$$

The value of $\Sigma \left(\frac{m}{r} \right)$ for the part KB may be deduced from this by using the angle OBA instead of OAB , that is, by writing $\frac{\pi}{2} - \theta$ for θ ; and therefore it is

$$\begin{aligned}
 &= \frac{M}{2a} \cdot \log_e \cot \left(\frac{\pi}{4} - \frac{\theta}{2} \right); \\
 \therefore \Sigma \left(\frac{m}{r} \right) &= \frac{M}{2a} \cdot \log_e \left\{ \cot \frac{\theta}{2} \cdot \cot \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right\};
 \end{aligned}$$

$$\begin{aligned}
 \therefore \frac{4}{3} M a^2 (d_t \theta)^2 &= C' - 2gMa \sin \theta \\
 &\quad + \frac{1}{2} Mga \log_e \left\{ \cot \frac{\theta}{2} \cot \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right\}.
 \end{aligned}$$

At the beginning of the motion $d_t \theta = 0$, and $\theta = \alpha$;

$$\therefore 0 = C' - 2gMa \sin \alpha + \frac{1}{2} Mga \log_e \left\{ \cot \frac{\alpha}{2} \cot \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) \right\};$$

$$\therefore \frac{4}{3} a (d_t \theta)^2 = 2g(\sin \alpha - \sin \theta) + \frac{1}{2} g \log_e \frac{\tan \frac{\alpha}{2} \tan \left(\frac{\pi}{4} - \frac{\alpha}{2} \right)}{\tan \frac{\theta}{2} \tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right)}.$$

From which the angular velocity is known when the rod is in any given position.

16. *Two weights connected by a flexible string without weight, hang over a fixed pulley, which is sufficiently rough to prevent the string sliding. To determine the motion.*

Let m, m', M be the masses of the weights and pulley, the former descending. Mk^2 the moment of inertia of the pulley about its axis. The equation of Vis Viva applies, because the friction between the string and pulley is *rolling* friction.

$x = Am$, $x' = Am'$, a = length of the string, b = radius of the pulley, θ = the angle through which the pulley has turned while m descends from A to its present position;

$$\therefore x = b\theta, \text{ and } x = a - x';$$

$$\text{and } \therefore d_t x = b d_t \theta, \text{ and } d_t x = -d_t x'.$$

Now, $\Sigma(mv^2) = m(d_t x)^2 + m'(d_t x')^2 + Mk^2(d_t \theta)^2 \dots \text{Art. 203,}$

$$= \left(m + m' + M \frac{k^2}{b^2} \right) (d_t x)^2.$$

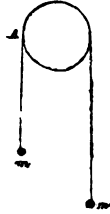
$$\begin{aligned} \text{And } \Sigma \{ m(Xdx + Ydy + Zdz) \} &= mgdx + m'gd x' \\ &= (m - m')gdx; \end{aligned}$$

$$\therefore \Sigma \{ m f(Xdx + Ydy + Zdz) \} = (m - m')g(x + C);$$

$$\therefore \left(m + m' + M \frac{k^2}{b^2} \right) (d_t x)^2 = 2(m - m')g(x + C).$$

But when the motion begins $x = 0$, $d_t x = 0$; and $\therefore C = 0$;

$$\therefore \left(m + m' + M \frac{k^2}{b^2} \right) (d_t x)^2 = 2(m - m')gx \dots (1).$$



The student will find no difficulty in completing the solution.

If the tensions of the two parts of the string on opposite sides of the pulley should be required, we must employ the equations of motion. Thus let T be the tension of Am and T' the tension of the other part. Then the equations of motion for m and m' are

$$d_t^2 x = g - \frac{T}{m}, \quad \text{and} \quad d_t^2 x' = g - \frac{T'}{m'};$$

$$\therefore T = mg - m d_t^2 x,$$

$$\begin{aligned} \text{and } T' &= m'g - m' d_t^2 x' \\ &= m'g + m' d_t^2 x. \end{aligned}$$

In which we have only to substitute for $d_t^2 x$ its value obtained by differentiating equation (1).

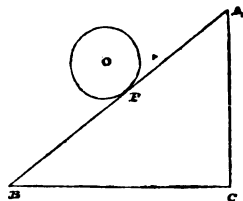
The pressure on the axis of the pulley

$$\begin{aligned} &= Mg + T + T' \\ &= (M + m + m')g - (m - m')d_t^2 x. \end{aligned}$$

17. *A cylinder rolls down an inclined plane, the friction of which is sufficient to prevent sliding; to determine the motion.*

Let ABC be the inclined plane, $\alpha = \angle ABC$; m = mass of the cylinder, a = its radius, mk^2 = its moment of inertia about its axis = $m \cdot \frac{a^2}{2}$.

$x = AP$ the space described by the centre of gravity of the cylinder,



θ = the angle through which the cylinder has turned about its axis since leaving A ;

$$\therefore x = a\theta, \quad \text{and} \quad d_t x = a d_t \theta.$$

Since the cylinder *rolls without* sliding, the equation of Vis Viva holds.

$$\begin{aligned}\text{Now } \Sigma (mv^2) &= m (d_t x)^2 + m k^2 (d_t \theta)^2 \\ &= m \left(1 + \frac{k^2}{a^2}\right) (d_t x)^2.\end{aligned}$$

$$\text{Also } \Sigma \{m (X dx + Y dy + Z dz)\} = mgd (x \sin \alpha);$$

$$\therefore m \left(1 + \frac{k^2}{a^2}\right) (d_t x)^2 = 2mg (x \sin \alpha + C).$$

But at the beginning of the motion $d_t x = 0$, $x = 0$, and
 $\therefore C = 0$;

$$\therefore \left(1 + \frac{k^2}{a^2}\right) (d_t x)^2 = 2gx \sin \alpha.$$

The student can complete the solution.

If the magnitude of the friction necessary to prevent sliding be required, denote it by F and employ the equations of motion. By the principles of translation and rotation, we may suppose $\frac{F}{m}$ and $g \sin \alpha$ to be applied at O the centre of gravity of the cylinder; and since O has described the same space as P by the nature of the system, therefore for the motion of O we have

$$d_t^2 x = g \sin \alpha - \frac{F}{m}.$$

And for the motion of rotation, supposing O a fixed point, we have

$$d_t^2 \theta = \frac{Fa}{mk^2};$$

$$\begin{aligned}\therefore d_t^2 x &= g \sin \alpha - \frac{k^2}{a} \cdot d_t^2 \theta \\ &= g \sin \alpha - \frac{k^2}{a^2} \cdot d_t^2 x;\end{aligned}$$

$$\therefore d_t^2 x = \frac{g \sin \alpha}{1 + \frac{k^2}{a^2}},$$

which leads to the same result as before.

$$F = mg \sin \alpha - m d_t^2 x$$

$$= mg \cdot \frac{k^2 \sin \alpha}{a^2 + k^2}.$$

If μ be the coefficient of sliding friction, the sliding friction $= \mu \cdot$ pressure. Now the pressure $= mg \cos \alpha$, consequently the sliding friction $= \mu \cdot mg \cos \alpha$. The body will consequently not wholly roll unless

$$\mu \cdot mg \cos \alpha \text{ be } > mg \cdot \frac{k^2 \sin \alpha}{a^2 + k^2},$$

$$\text{or } \tan \alpha < \mu \left(1 + \frac{a^2}{k^2} \right),$$

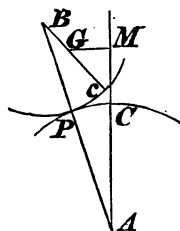
$$< 3\mu,$$

unless this condition be satisfied the body will partly roll and partly slide.

18. *A body, terminated at its lower extremity in a curve surface, rests on another curve surface in a position of stable equilibrium, and is slightly disturbed; to find the time of titubation.*

The time of titubation is the time of making a small oscillation.

Let the plane of the paper passing through G the centre of gravity of the titubating body be the plane of titubation; PC, Pc the sections of the surfaces made by it: A, B the centres of curvature of PC, Pc : C, c points which are in contact when the body is in a position of equilibrium; draw GM perpendicular to AC , which is vertical.



θ = the inclination of Bc to the vertical or AC produced,

m = mass of the titubating body,

mk^2 = its moment of inertia about G ,

$R = AC$, $r = Bc$, $Gc = h$, $\bar{x} = AM$, $\bar{y} = MG$.

Then $R \cdot \angle CAP = PC = Pc = r \cdot \angle PBC$.

But $\theta = CAP + PBC$;

$$\therefore PAC = \frac{r\theta}{R+r}.$$

Now $\bar{x} = AB \cos BAC - BG \cos \theta$

$$= (R+r) \cos \left(\frac{r\theta}{R+r} \right) - (r-h) \cos \theta$$

$$= (R+r) \left\{ 1 - \frac{1}{2} \cdot \frac{r^2 \theta^2}{(R+r)^2} \right\} - (r-h) \left(1 - \frac{1}{2} \theta^2 \right),$$

by expanding and neglecting $\theta^4, \theta^6, \dots$

$$= R+h + \left(\frac{Rr}{R+r} - h \right) \frac{\theta^2}{2}.$$

If the greatest value of θ be α , the value of $\bar{x}$ at the beginning of the titubation

$$= R+h + \left(\frac{Rr}{R+r} - h \right) \frac{\alpha^2}{2}.$$

Hence the space fallen through by G

$$= \frac{1}{2} \left(\frac{Rr}{R+r} - h \right) (\alpha^2 - \theta^2).$$

Now because the body rolls without sliding, the equation of Vis Viva holds;

$$\therefore 2 \Sigma \{ m \int (X dx + \&c.) \} = mg \left(\frac{Rr}{R+r} - h \right) (\alpha^2 - \theta^2) \dots \text{Art. 203.}$$

$$\text{But } \Sigma (mv^2) = m \{ (d_t \bar{x})^2 + (d_t \bar{y})^2 \} + mk^2 (d_t \theta)^2,$$

$$\text{and } d_t \bar{x} = \left(\frac{Rr}{R+r} - h \right) \theta d_t \theta,$$

the square of which may be neglected, being of a higher order than θ^2 .

$$y = AB \sin BAC - BG \sin \theta$$

$$= (R + r) \cdot \frac{r\theta}{R + r} - (r - h) \theta = h\theta;$$

$$\therefore d_t \bar{y} = h d_t \theta;$$

$$\therefore \Sigma (mv^2) = mh^2 (d_t \theta)^2 + mk^2 (d_t \theta)^2;$$

$$\therefore (h^2 + k^2) (d_t \theta)^2 = g \left(\frac{Rr}{R + r} - h \right) (\alpha^2 - \theta^2);$$

This being differentiated and divided by $2(h^2 + k^2) d_t \theta$ gives

$$d_t^2 \theta + g \frac{\frac{Rr}{R + r} - h}{h^2 + k^2} \theta = 0.$$

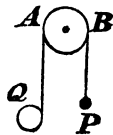
This being of the form pointed out in Art. 152 as the equation of vibration, we know at once that the time of titubation

$$= \frac{\pi}{\sqrt{g}} \cdot \left(\frac{h^2 + k^2}{\frac{Rr}{R + r} - h} \right)^{\frac{1}{2}}.$$

Wherefore the length of a simple pendulum which would oscillate in the same time is $\frac{h^2 + k^2}{\frac{Rr}{R + r} - h}$.

19. *To one end of a string which passes over a rough fixed pulley is fastened a weight, the other end is wrapped many times round a heavy cylinder. Determine the motion when the cylinder and weight are permitted to descend.*

Let P , Q be the masses of the weight and cylinder, M the mass of the fixed pulley, b = its radius, a = the radius of the cylinder: T the tension of BP , T' that of AQ : $x = BP$, $x' = AQ$, and θ , ϕ the angles through which the cylinder and pulley have revolved, in the time t from the beginning of the motion.



For the motion of P , we have

$$d_t^2 x = g - \frac{T}{P} \dots \dots \dots (1).$$

For the descent of the centre of gravity of the cylinder

$$d_t^2 x' = g - \frac{T'}{Q} \dots \dots \dots (2).$$

For its angular motion

$$d_t^2 \theta = \frac{T' a}{Q \frac{a^2}{2}} \dots \dots \dots (3).$$

For the angular motion of the pulley

$$d_t^2 \phi = \frac{(T - T') b}{M \frac{b^2}{2}} \dots \dots \dots (4).$$

These are all the equations of motion; but there are geometrical conditions to which the variables are subject. Thus, the velocity of P must be equal to the velocity of the circumference of the pulley; which property furnishes the equation

$$d_t x = b d_t \phi :$$

also, as the string is unwrapped from the cylinder it is employed in increasing the lengths of BP and AQ ; and therefore the velocity of the circumference of the cylinder arising from its angular motion is equal to the sum of the velocities of P and Q , that is,

$$a d_t \theta = d_t x + d_t x'.$$

This and the preceding equations being differentiated, give

$$d_t^2 x = b d_t^2 \phi \dots \dots \dots (5),$$

$$a d_t^2 \theta = d_t^2 x + d_t^2 x' \dots \dots \dots (6).$$

Substituting in (5) from (1) and (4), we have

$$g - \frac{T}{P} = \frac{2(T - T')}{M};$$

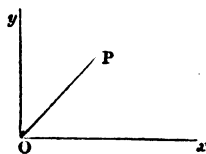
and substituting in (6) from (3) (1) and (2), we have

$$2g = \frac{T}{P} + \frac{T'}{Q}.$$

These two equations furnish the values of T and T' , and shew that they are constant; being substituted in equations (1) (2) (3) (4), we see that the descent of both P and Q is uniformly accelerated; as are also the angular motions of the pulley and cylinder. With this help the student will be able to complete the solution.

20. *On a rough horizontal plane revolving uniformly about a vertical axis a rough sphere is placed: determine its initial motions, and shew that its path in space will be a circle.*

Let O be the point about which the plane revolves; Ox , Oy co-ordinate axes fixed in space; P the place of contact of the ball at the time t ; x , y its co-ordinates. Suppose the plane to revolve from x towards y .



F , G the friction parallel to Ox and Oy respectively.

ω = the angular velocity of the plane.

a = the radius of the ball, m = its mass,

mk^2 = its moment of inertia about a diameter,

$r = OP$.

For the motion of the centre of gravity of the ball we have

$$d_t^2 x = \frac{F}{m}, \quad d_t^2 y = \frac{G}{m}.$$

To determine the angular motion of the ball, we may suppose its centre of gravity fixed. If then ω_1 , ω_2 , be its angular velocities about axes through its centre parallel to Ox and Oy , the former being estimated in such a direction as would elevate Oy above the paper, and the latter to elevate Ox above the paper: then

$$\left. \begin{aligned} d_t \omega_1 &= \frac{G \cdot a}{m k^2} \\ \text{and } d_t \omega_2 &= \frac{F \cdot a}{m k^2} \end{aligned} \right\} \dots\dots \text{Art. 252 ;}$$

$$\therefore d_t^2 x = \frac{k^2}{a} d_t \omega_2, \quad d_t^2 y = \frac{k^2}{a} d_t \omega_1 ;$$

$$\therefore d_t x = \frac{k^2}{a} (\omega + C), \quad d_t y = \frac{k^2}{a} (\omega_1 + C').$$

Now the linear velocity of the centre of gravity of the ball due to the rotation of the ball is

$$- a \omega, \text{ parallel to } Ox,$$

$$- a \omega_1 \dots\dots\dots Oy,$$

and the linear velocity of the centre of gravity due to the rotation of the plane, is

$$- \omega y \text{ parallel to } Ox,$$

$$\omega x \dots\dots\dots Oy ;$$

$$\therefore d_t x = - \omega y - a \omega_2,$$

$$\text{and } d_t y = \omega x - a \omega_1,$$

or, by eliminating ω_1 and ω_2 ,

$$d_t x = - \omega y - a \left(\frac{a d_t x}{k^2} - C \right),$$

$$d_t y = \omega x - a \left(\frac{a d_t y}{k^2} - C' \right) ;$$

$$\therefore \left(1 + \frac{a^2}{k^2} \right) d_t x = a C - \omega y,$$

$$\text{and } \left(1 + \frac{a^2}{k^2} \right) d_t y = a C' + \omega x ;$$

eliminating t , we have

$$d_x y = \frac{a C' + \omega x}{a C - \omega y} ;$$

$$\therefore 2aCy - \omega y^2 = 2aCx + \omega x^2 + C'',$$

$$\text{or } x^2 + y^2 + \frac{2aC'}{\omega}x - \frac{2aC}{\omega}y + \frac{C''}{\omega} = 0,$$

which is the equation of a circle.

We have to determine the values of the constants C , C' , C'' ; to do this we must consider the initial circumstances of the motion. Let the axis of x be in such a position as to pass through the point where the ball was first laid upon the plane; and let the distance of this point from O be b ; at that moment $d_t x = 0$, $\omega_2 = 0$, and therefore $C = 0$. To find C' we must consider that the roughness of the plane is such as to admit of no sliding, and consequently the instant the ball touches the plane, it is put into such a state of motion as enables the combined motions of rotation and translation to keep pace with the plane. Consequently, the initial friction of the plane strikes the ball a blow parallel to the axis of y . Let this blow be I ; this being applied to the centre of gravity, gives a velocity parallel to the axis of $y = \frac{I}{m}$ (Art. 34); and an angular velocity about an axis through the centre parallel to $Ox = \frac{Ia}{mk^2}$ (Art. 253).

Hence, as before

$$\frac{I}{m} = \omega b - \frac{Ia^2}{mk^2};$$

$$\therefore \frac{I}{m} = \frac{\omega b}{1 + \frac{a^2}{k^2}};$$

$$\therefore \text{angular velocity} = \frac{Ia}{mk^2} = \frac{\omega ab}{a^2 + k^2}.$$

This is the first value of ω_1 , and the first value of $d_t y$ is $\frac{I}{m}$;

$$\therefore \frac{\omega b}{1 + \frac{k^2}{a^2}} = \frac{k^2}{a} \left(\frac{\omega ab}{a^2 + k^2} + C' \right);$$

$$\therefore C' = 0;$$

$$\therefore x^2 + y^2 + \frac{C''}{\omega} = 0.$$

$$\text{But when } y = 0, x = b; \therefore C'' = -\omega b^2;$$

$$\therefore x^2 + y^2 = b^2,$$

which is the equation of a circle whose centre is O .

The initial motions are known from what has been done above; for the instant the sphere touches the plane, the plane communicates to its centre of gravity a velocity, parallel to Oy , equal to

$$\frac{I}{m} \text{ or } \frac{\omega b k^2}{a^2 + k^2},$$

and gives it an angular velocity equal to

$$\frac{Ia}{mk^2} \text{ or } \frac{\omega ab}{a^2 + k^2},$$

about an axis parallel to Ox .

21. *A rough body is placed with its flat surface upon a rough board, and this lies upon a smooth horizontal plane. The board is projected along the plane with a given velocity. Determine the motion of the body and board.*

For simplicity, we shall suppose the vertical plane which passes through the centres of gravity of the body and board to be parallel to the line of projection.

Let OA be the intersection of this vertical plane with



the horizontal plane. It will simplify our equations without diminishing the generality of our investigation, if we suppose G, G' the centres of gravity of the board and the body to have been at the moment of projection in the same vertical line; let this line meet the plane in O .

$x = OG$, $x' = OG'$, u = velocity of projection,

m = mass of the plane, m' = mass of the body,

μ = coefficient of sliding friction between the body and board.

Then the friction $= \mu m'g$, which retards the board and accelerates the body. The retarding force upon the board is $\frac{\mu m'g}{m}$ which being constant, we have

$$x = ut - \frac{\mu m'g}{2m} t^2 \dots \dots \text{Art. 62;}$$

$$\text{and } \therefore d_1x = u - \frac{\mu m'g}{m} t.$$

The accelerating force upon the body is μg , which being also constant, we have

$$x' = \frac{\mu g}{2} \cdot t^2 \dots \dots \text{Art. 59;}$$

$$\text{and } \therefore d_1x' = \mu g t.$$

The sliding friction acts as long as the board moves faster than the body, and ceases the instant these are equal: the time when this happens will therefore be found from the equation $d_1x = d_1x'$, or

$$u - \frac{\mu m'g}{m} t = \mu g t;$$

$$\therefore t = \frac{mu}{\mu g(m + m')}.$$

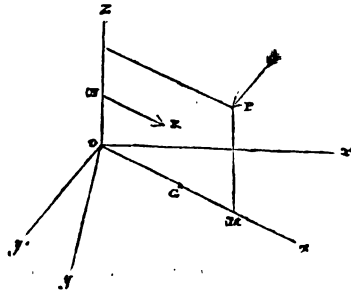
$$\text{At this moment } d_1x' = \mu g t = \frac{mu}{m + m'}.$$

With this velocity the board and body will go on together for ever: the space described before the system attains this state

$$= \frac{1}{2} \mu g t^2 = \left(\frac{m}{m + m'} \right)^2 \cdot \frac{u^2}{2 \mu g}.$$

22. *A body revolves round an axis by the action of a constant pressure in a direction always perpendicular to a plane passing through the axis and the centre of gravity. Determine the position of the point of application of the force, that there may be no pressure on the axis except in this plane.*

Let the axis of revolution be taken for the axis of z , and suppose the system to be referred to a set of co-ordinate axes Ox', Oy', Oz fixed in the body, as well as to the system Ox, Oy, Oz fixed in space; the planes $xy, x'y'$ passing through G the centre of gravity of the body, and the plane $x'z$ being that to which the given pressure P is always perpendicular.



This pressure we may suppose applied perpendicularly at P a point of the body which is in the plane $x'z$.

$OG = a$, $OM = h$, $MP = k$, the co-ordinates of P at the time t ; M = the mass of the body.

The pressure on the axis can always be reduced to a single force and a couple, but in the present case, since they all act in one plane $x'z$ by the question, they can be reduced to a single force R acting at some point N parallel to Ox' .

$l = ON$, $\theta = xOx'$; x, y, z the co-ordinates of any particle whose mass is m , referred to the axes in space; x', y', z the co-ordinates of the same particle referred to the axes in the body. Then for the motion of translation,

$$d_t^2 \bar{x} = -\frac{P}{M} \sin \theta + \frac{R}{M} \cos \theta \dots\dots(1),$$

$$d_t^2 \bar{y} = \frac{P}{M} \cos \theta + \frac{R}{M} \sin \theta \dots\dots\dots(2),$$

$$d_t^2 \bar{z} = 0.$$

And for the motion of rotation

$$\Sigma \{m (y d_t^2 x - x d_t^2 y)\} = -P \cos \theta . k - R \sin \theta . l,$$

$$\Sigma \{m (x d_t^2 x - x d_t^2 x)\} = -P \sin \theta . k + R \cos \theta . l,$$

$$\Sigma \{m (x d_t^2 y - y d_t^2 x)\} = Ph.$$

By the nature of the case x is constant for the same particle, and therefore $d_t^2 x = 0$, hence the equations for the rotation are reduced to

$$\Sigma (m x d_t^2 y) = P k \cos \theta + R l \sin \theta \dots\dots\dots(3),$$

$$\Sigma (m x d_t^2 x) = -P k \sin \theta + R l \cos \theta \dots\dots\dots(4),$$

$$\Sigma \{m (x d_t^2 y - y d_t^2 x)\} = Ph \dots\dots\dots(5).$$

Now $\bar{x} = a \cos \theta$ and $\bar{y} = a \sin \theta$, which being written in equations (1) (2), give

$$a \sin \theta d_t^2 \theta + a \cos \theta (d_t \theta)^2 = \frac{P}{M} \sin \theta - \frac{R}{M} \cos \theta \dots\dots\dots(6),$$

$$\text{and } a \cos \theta d_t^2 \theta - a \sin \theta (d_t \theta)^2 = \frac{P}{M} \cos \theta + \frac{R}{M} \sin \theta \dots\dots\dots(7),$$

between which eliminating R , which is unknown, we have

$$\left. \begin{aligned} a d_t^2 \theta &= \frac{P}{M} \\ \therefore a d_t \theta &= \frac{P}{M} . t \end{aligned} \right\} \dots\dots\dots(8).$$

Again,

$$\Sigma (m x d_t^2 y) = d_t^2 \Sigma (m x y) \text{ and } \Sigma (m x d_t^2 x) = d_t^2 \Sigma (m x x).$$

$$\text{But } x = x' \cos \theta - y' \sin \theta,$$

$$\text{and } y = x' \sin \theta + y' \cos \theta ;$$

$$\begin{aligned} \therefore \Sigma (m x d_t^2 y) &= d_t^2 \Sigma \{m x (x' \sin \theta + y' \cos \theta)\} \\ &= \Sigma (m x x') . d_t^2 \sin \theta + \Sigma (m x y') . d_t^2 \cos \theta \\ &= G d_t^2 \sin \theta + F d_t^2 \cos \theta. \end{aligned}$$

$$\text{So } \Sigma (m x d_t^2 x) = G d_t^2 \cos \theta - F d_t^2 \sin \theta.$$

$$\text{Also } \Sigma \{m(xd_t y - yd_t x)\} = \Sigma (mx'^2) d_t \theta = fd_t \theta;$$

$$\therefore \Sigma \{m(xd_t^2 y - yd_t^2 x)\} = fd_t^2 \theta.$$

Wherefore equation (5) becomes

$$d_t^2 \theta = \frac{Ph}{f};$$

$$\therefore h = \frac{f}{aM} = \frac{\Sigma(mx'^2)}{aM} \text{ from (8).}$$

Eliminating R between (3) and (4), we obtain

$$\begin{aligned} Pk &= \Sigma (mxd_t^2 y) \cos \theta - \Sigma (mxd_t^2 x) \sin \theta \\ &= G (\cos \theta d_t^2 \sin \theta - \sin \theta d_t^2 \cos \theta) \\ &\quad + F (\cos \theta d_t^2 \cos \theta + \sin \theta d_t^2 \sin \theta) \\ &= G d_t^2 \theta - F (d_t \theta)^2 \end{aligned}$$

$$= \frac{GP}{aM} - \frac{FP^2}{a^2 M^2} t^2, \text{ from (8);}$$

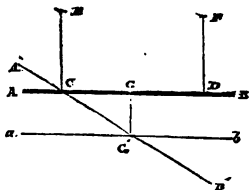
$$\therefore k = \frac{\Sigma (mx'x')}{aM} - \frac{P\Sigma (mx'y')}{a^2 M^2} t^2.$$

Hence h is constant; and P descends in the straight line PM according to the law of a particle acted on by a constant accelerating force.

23. *A straight rod is suspended in a horizontal position by two parallel strings fastened to it at given points. Compare the tension of one of them with its initial tension when the other suddenly breaks, or is cut in two.*

Let AB be the rod, and CE , DF the strings, at first. M = mass of the rod, Mk^2 its moment of inertia about its centre of gravity G . $CG = a$, $DG = b$. Therefore the tension of CE is $\frac{bMg}{a+b}$ before the breaking

of FD . When FD breaks, let the initial tension of CE be T ; and suppose the centre of gravity G to descend through $GG' = s$ in



the time t ; and the rod to turn through the angle $B'CG$ or $aG'A' = \theta$ in the same time;

$$\therefore s = a\theta \quad \text{and} \quad d_t^2 s = a d_t^2 \theta.$$

By the principles of Arts. 196, 197, we may consider the motion of translation and rotation separately. By the former principle the body would descend into the position ab ; and by the latter it would, G' being fixed, revolve into the position $A'B'$.

For the motion of G , we have

$$d_t^2 s = g - \frac{T}{M},$$

and for the angular motion about G' , we have

$$d_t^2 \theta = \frac{Ta}{Mk^2} \dots\dots \text{Art. 252};$$

$$\therefore g - \frac{T}{M} = \frac{Ta^2}{Mk^2};$$

$$\therefore T = \frac{Mgk^2}{k^2 + a^2};$$

$$\therefore \frac{\text{tension before breaking}}{\text{tension after breaking}} = \frac{b}{a+b} \cdot \frac{k^2 + a^2}{k^2}.$$

24. *A free quiescent body is struck by an impulsive force: to find the axis about which it will begin to revolve.*

This is called the axis of *spontaneous rotation*.

Let a plane be drawn in the body through the centre of gravity; and before the blow is given let this plane be perpendicular to the direction of the impulse I : let P be the point in which the direction of the impulse meets the plane: we may suppose I communicated to the body immediately at this point. Take initial position of G as origin of co-ordinates, and initial position of GP as axis of x , that of y being parallel to I : put $GP = x'$, m = the mass of the body, mk^2 = its moment of inertia about axis of x .

For the motion of the centre of gravity, we have

$$d_t \bar{y} = \frac{I}{m} \dots \text{Art. 220.}$$

Also if $\omega_1, \omega_2, \omega_3$ be the angular velocities about the axes of x, y, z , we have (by Arts. 200, 253)

$$\omega_1 = \frac{\text{mom. of } I \text{ about axis of } x}{\text{mom. inert. about axis of } x} = 0,$$

$$\omega_2 = \frac{\text{mom. of } I \text{ about axis of } y}{\text{mom. inert. about axis of } y} = 0,$$

$$\omega_3 = \frac{\text{mom. of } I \text{ about axis of } z}{\text{mom. inert. about axis of } z} = \frac{I x'}{m k^2}.$$

Thus we see that the angular motion takes place about an axis parallel to that of z . Also because G has no motion parallel to the plane xz , the axis of spontaneous rotation must be in that plane. Let Q be the point where it meets the axis of x , and put $GQ = x$. Then the velocity of Q

$$\text{from translation of } G = \frac{I}{m},$$

$$\text{from rotation about } G = \omega_3 x = \frac{I x' x}{m k^2};$$

$$\therefore \text{ whole vel. of } Q = \frac{I}{m} + \frac{I x x'}{m k^2}.$$

But by hypothesis Q is in the axis of spontaneous rotation, and therefore has no motion,

$$\therefore 0 = \frac{I}{m} + \frac{I x x'}{m k^2};$$

$$\therefore x = -\frac{k^2}{x'};$$

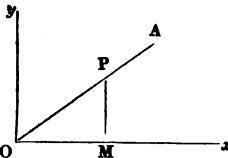
$$\text{or } QG \cdot GP = k^2.$$

The negative sign shews that the centre of percussion and axis of spontaneous rotation are on opposite sides of the centre of gravity. This determines the position of the required axis:

and compared with Art. 268 shews that the centre of percussion and axis of spontaneous rotation have the same relation to each other as the centre of oscillation and axis of suspension.

25. *A straight rod, moveable in a vertical plane, about a hinge at one end, rests on a peg in a horizontal position. By what vertical impulse must it be struck underneath that it may just rise into a vertical position?*

Let OA be the rod in any position of its ascent, O being the hinge; let Ox , Oy , the axes of co-ordinates be horizontal and vertical. M = mass of the rod, Mk'^2 its moment of inertia about O , m = mass of a particle at P , the co-ordinates of which are x , y . $OP = r$, $AOx = \theta$, $OA = a$. Then, since the only impressed force in this position of the rod is gravity, the equation for determining the rotation about O is



$$\begin{aligned} d_t^2\theta &= -\frac{\Sigma(mgx)}{\Sigma(mr^2)} \dots\dots \text{Art 252} \\ &= -\frac{Mg\bar{x}}{Mk'^2} = -\frac{ga \cos \theta}{2k'^2}. \end{aligned}$$

Multiply by $2d_t\theta$, and integrate;

$$\therefore (d_t\theta)^2 = C - \frac{ga}{k'^2} \sin \theta.$$

But if ω be the angular velocity communicated by the impulse, $d_t\theta = \omega$ when $\theta = 0$, and therefore $C = \omega^2$;

$$\therefore (d_t\theta)^2 = \omega^2 - \frac{ga}{k'^2} \sin \theta.$$

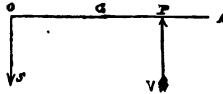
And by the question $d_t\theta = 0$ when $\theta = \frac{\pi}{2}$;

$$\therefore \omega^2 = \frac{ga}{k'^2}.$$

Wherefore the impulse must be such as will communicate this angular velocity to the rod.

Let now OA be the beam in a horizontal position, and let it be struck at P by an impulse I . Then, since ω is the angular velocity produced by this action, we have by Art. 263.

$$\omega = \frac{I \cdot OP}{\text{mom. inert. about } O},$$

$$\text{or } \frac{\sqrt{ga}}{k'} = \frac{I \cdot OP}{Mk'^2};$$


$$\therefore I = M \sqrt{ag} \cdot \frac{k'}{OP}$$

$$= M \left(\frac{ag}{3} \right)^{\frac{1}{2}} \cdot \frac{OA}{OP}, \quad \because k'^2 = \frac{1}{3} OA^2.$$

This gives the value of the impulse I .

If the impulsive stress on the hinge be required, it may be found thus :

Call it S . Then if we apply it to the body at O in a direction opposite to that in which I acts, we may consider the body as free. Hence, for the motion of G , we have

$$OG \cdot \omega = \text{velocity of } G$$

$$= \frac{I - S}{M} \dots \dots \text{Art. 253;}$$

$$\therefore S = I - M \cdot OG \cdot \omega$$

$$= M \cdot \left(\frac{ag}{3} \right)^{\frac{1}{2}} \cdot \left(\frac{OA}{OP} - \frac{3OG}{OA} \right).$$

26. PROB. *A circular hoop has communicated to it a velocity of translation, parallel to a given rough inclined plane with which it is in contact, and in the downward direction, and also at the same time a velocity of rotation about its centre. Find the conditions under which the hoop will descend to a given point on the inclined plane, and then just return to the place where it set out.*

Let m = mass of the hoop, a its radius: α = inclination of the plane to the horizon, μ = coefficient of friction, g = force of gravity, x = distance (at the time t) of the point of contact of the hoop and plane from the starting point: u = the given velocity of translation, ω = initial velocity of rotation.

(The translation of the hoop is continually retarded by friction, which at length destroys all the initial velocity, and causes the hoop to return up the plane; the acceleration of the velocity of return by friction continues as long as the point of contact has a sliding motion, and ceases when the sliding ceases; from that moment the hoop rolls up the plane till gravity destroys the velocity of translation up the plane which friction generated.)

The pressure of the hoop on the plane = $mg \cos \alpha$;

$\therefore$ sliding friction = $\mu mg \cos \alpha$;

$\therefore$ retarding force of friction on the motion of translation

$$= \mu g \cos \alpha;$$

$$\text{hence } d_t^2 x = g \sin \alpha - \mu g \cos \alpha \dots\dots(1).$$

This equation holds until the sliding ceases.

Again, the angular motion is retarded by $\mu mg \cos \alpha$ acting at the circumference of the hoop; if therefore θ be the angle through which the hoop has revolved in the time t ,

$$d_t^2 \theta = - \frac{\mu mg \cos \alpha \cdot a}{ma^2} = - \frac{\mu g \cos \alpha}{a} \dots\dots(2).$$

This equation holds also until the sliding ceases.

Now suppose the sliding to last for the time τ : and integrate (1) and (2);

$$\therefore d_t x = g (\sin \alpha - \mu \cos \alpha) t + u \dots\dots(3);$$

(the constant = u , $\therefore$ when $t = 0$, $d_t x = u$) :

$$\text{also } d_t \theta = - \frac{\mu g \cos \alpha}{a} t + \omega \dots\dots\dots(4);$$

(the constant = ω , $\therefore$ when $t = 0$, then $d_t \theta = \omega$).

When $t = \tau$ there is no sliding at the point of contact, and

$$\therefore 0 = d_t x + a d_t \theta ;$$

$$\therefore 0 = g (\sin \alpha - \mu \cos \alpha) \tau + u - \mu g \tau \cos \alpha + a \omega ;$$

$$\therefore \tau = \frac{u + a \omega}{(2\mu \cos \alpha - \sin \alpha)g} \dots\dots\dots(5).$$

The velocity of translation at the time τ , obtained from (3), is

$$u + g (\sin \alpha - \mu \cos \alpha) \tau \dots\dots\dots(6).$$

Now (3) being integrated gives

$$x = \frac{1}{2} g t^2 (\sin \alpha - \mu \cos \alpha) + u t \dots\dots\dots(7).$$

No constant is added, because $x = 0$ when $t = 0$. Let x' be the value of x at the time τ ,

$$\therefore x' = \frac{1}{2} g \tau^2 (\sin \alpha - \mu \cos \alpha) + u \tau \dots\dots\dots(8).$$

After the time τ the friction is not sliding friction, but is just sufficient to prevent sliding; call it F , then the equations of motion are

$$d_t^2 x = g \sin \alpha - \frac{F}{m} \text{ and } d_t^2 \theta = - \frac{F a}{m a^2}.$$

But since there is now no sliding of the point of contact,

$$\therefore \text{always } 0 = d_t x + a d_t \theta, \text{ and } \therefore 0 = d_t^2 x + a d_t^2 \theta = g \sin \alpha - \frac{2F}{m};$$

$$\therefore \frac{F}{m} = \frac{1}{2} g \sin \alpha,$$

and by substitution,

$$d_t^2 x = \frac{1}{2} g \sin \alpha ;$$

wherefore by integration,

$$d_t x = \frac{1}{2} g t \sin \alpha + C ;$$

but when $t = \tau$, $d_t x$ has the value marked (6),

$$\therefore C = u + \frac{1}{2} g \tau (\sin \alpha - 2\mu \cos \alpha),$$

and by substitution,

$$d_t x = \frac{1}{2} g t \sin \alpha + u + \frac{1}{2} g \tau (\sin \alpha - 2\mu \cos \alpha) \dots\dots(9).$$

Integrating this,

$$x = \frac{1}{4}gt^2 \sin \alpha + ut + \frac{1}{2}g\tau t (\sin \alpha - 2\mu \cos \alpha) + C.$$

But when $t = \tau$, $x = x'$; hence from (8),

$$\begin{aligned} \frac{1}{2}g\tau^2 (\sin \alpha - \mu \cos \alpha) + u\tau = \frac{1}{4}g\tau^2 \sin \alpha + u\tau \\ + \frac{1}{2}g\tau^2 (\sin \alpha - 2\mu \cos \alpha) + C, \end{aligned}$$

$$\text{and } C = -\frac{1}{4}g\tau^2 (\sin \alpha - 2\mu \cos \alpha);$$

$$\therefore x = \frac{1}{4}gt^2 \sin \alpha + ut + \frac{1}{2}g\tau (t - \frac{1}{2}\tau) (\sin \alpha - 2\mu \cos \alpha) \dots (10).$$

Now the same value of t (the time, from the beginning, when the motion ceases and the body has returned to the point of projection) which makes $x = 0$, also makes $d_t x = 0$; and hence (9) and (10) are satisfied together when their left-hand members vanish:

$$\therefore 0 = gt \sin \alpha + 2u + g\tau (\sin \alpha - 2\mu \cos \alpha) \dots (11),$$

$$\text{and } 0 = \frac{1}{2}gt^2 \sin \alpha + 2ut + g\tau (t - \frac{1}{2}\tau) (\sin \alpha - 2\mu \cos \alpha) \dots (12),$$

(11) $\times t -$ (12) gives

$$0 = \frac{1}{2}gt^2 \sin \alpha + \frac{1}{2}g\tau^2 (\sin \alpha - 2\mu \cos \alpha),$$

$$\text{or } t = \tau (2\mu \cot \alpha - 1)^{\frac{1}{2}}.$$

This being written in (11), we obtain

$$\frac{2u}{\sin \alpha} = g\tau \{ (2\mu \cot \alpha - 1) - (2\mu \cot \alpha - 1)^{\frac{1}{2}} \};$$

$$\therefore 2u = (u + a\omega) \cdot \{1 - (2\mu \cot \alpha - 1)^{-\frac{1}{2}}\} \text{ from (5),}$$

$$\text{or } a\omega + u = (a\omega - u) (2\mu \cot \alpha - 1)^{\frac{1}{2}} \dots (13).$$

But by the question the hoop descended to a given point ($x = b$ suppose) before it began to return. Hence the same value of t which in (7) makes $x = b$, makes $d_t x = 0$ in (8). We must therefore eliminate t between the equations

$$0 = g (\sin \alpha - \mu \cos \alpha) t + u$$

$$\text{and } b = \frac{1}{2}gt^2 (\sin \alpha - \mu \cos \alpha) + ut.$$

Multiply the former of these by $\frac{1}{2}t$ and subtract from the latter,

$$\therefore b = \frac{1}{2}ut, \text{ or } t = \frac{2b}{u}.$$

This value of t is the time when the hoop began to return up the plane, and being written in the former of the equations from which it was obtained, it gives

$$0 = \frac{2bg}{u} (\sin \alpha - \mu \cos \alpha) + u;$$

$$\therefore 2\mu \cot \alpha - 1 = \frac{u^2}{bg \sin \alpha} + 1 \dots\dots (14);$$

and $\therefore$ from (13)

$$\left(\frac{a\omega + u}{a\omega - u} \right)^2 = 1 + \frac{u^2}{bg \sin \alpha}$$

$$\text{or, } (a\omega - u)^2 = \frac{4abg\omega}{u} \sin \alpha,$$

which expresses the relation between the initial velocities of translation and rotation; and being solved as a quadratic, it gives

$$(ua\omega)^{\frac{1}{2}} = (bg \sin \alpha)^{\frac{1}{2}} + (u^2 + bg \sin \alpha)^{\frac{1}{2}}.$$

With respect to the friction, (14) shews that

$$\mu \cot \alpha > 1,$$

$$\text{or } \mu > \tan \alpha:$$

and therefore the inclination of the plane must be less than that at which a body would begin to slide down by its own weight.

CHAPTER XI.

ON MOTION IN A RESISTING MEDIUM.

293. IN the preceding pages, the motion of a body has always been supposed to take place in a vacuum (see page 54). If a body move in a medium, it displaces the particles of the medium as it successively comes in contact with them. Momentum is thus communicated to them and lost by the body (Art. 27). Hence the resistance of a medium is of the nature of a *retarding* pressure, taking place in a direction opposite to the motion. If we reckon this pressure among the forces impressed upon the body, we may consider it as moving in a vacuum, and apply the methods employed in the preceding articles. It is found, by experiment, that the retarding force of a medium, for a given body, varies as (density) . (velocity)² when the velocity is neither very great nor very small. We shall, therefore, in general, represent it by kv^2 ; k being a quantity depending on the form of the body and the density of the medium, and v being the velocity of the body.

294. It is to be observed also, that if a body whose volume is V , and density ρ be within a medium whose density is ρ' , and be subject to the action of the force of gravity g ; then

pressure of the body downwards = $Vg\rho$,

..... medium upwards = $Vg\rho'$;

$\therefore$ moving force on the body downwards = $Vg(\rho - \rho')$,

and the mass of the moving body = $V\rho$;

$\therefore$ accelerating force on the body downwards

$$= g \left(1 - \frac{\rho'}{\rho} \right) \dots\dots \text{Art. 23.}$$

This quantity we shall always represent by g' ; it is called the force of relative gravity.

Notwithstanding the retarding force of a medium varies as the square of the velocity; we shall sometimes, for the sake of examples, suppose it to vary according to other laws.

295. *PROB. A particle acted on by no forces is projected in a uniform medium, the resistance of which varies as the velocity; to determine the motion.*

Let u = the velocity of projection, and s the space described in the time t . Then, since the resistance acts in a direction opposite to the motion, it will not make the body deviate from the direction in which it is at any time moving: consequently, the path will be a straight line. And the equation of motion is

$$\begin{aligned} d_t^2 s &= -k v, \\ &= -k d_t s; \\ \therefore d_t s &= C - k s. \end{aligned}$$

But when $s = 0$, $d_t s = u$; $\therefore C = u$,

and $d_t s = u - k s$;

$$\therefore k = \frac{k d_t s}{u - k s};$$

$$\therefore k t = -\log_e (u - k s) + C.$$

But when $t = 0$, $s = 0$, and $\therefore C = \log u$;

$$\therefore k t = \log_e \frac{u}{u - k s};$$

$$\therefore s = \frac{u}{k} \cdot (1 - e^{-kt}),$$

$$\text{and } d_t s = u e^{-kt}.$$

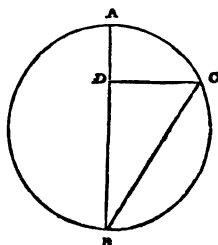
The last two equations determine the space described, and the velocity, at any given time.

296. *If chords be drawn from either extremity of a vertical diameter of a circle, the time of descent down each of them will be equal to the time down the diameter.*

Let g' be the relative gravity (Art. 294), $a = AB$, $a = ABC$, and let s = space descended from C down the chord CB in the time t ;

$$\therefore d_t^2 s = g' \cos \alpha - k d_t s ;$$

and $\therefore d_t s = g' \cos \alpha \cdot t - ks$, by integration, no constant being added because $d_t s$, t and s are zero together.



We may integrate the first equation otherwise; for by division

$$k = \frac{k d_t^2 s}{g' \cos \alpha - k d_t s} ;$$

$$\therefore kt = -\log_e (g' \cos \alpha - k d_t s) + C,$$

when $t = 0$, $d_t s = 0$; and $\therefore C = \log_e (g' \cos \alpha)$;

$$\therefore kt = \log_e \frac{g' \cos \alpha}{g' \cos \alpha - k d_t s} ;$$

$$\therefore g' \cos \alpha e^{-kt} = g' \cos \alpha - k d_t s$$

$$= g' \cos \alpha - k (g' t \cos \alpha - ks) ;$$

$$\therefore k^2 s = g' \cos \alpha (kt + e^{-kt} - 1).$$

When the body arrives at the bottom of the plane $s = a \cos \alpha$, and

$$\therefore k^2 a = g' (kt + e^{-kt} - 1).$$

Consequently t is constant; for being independent of α , it is the same for any one of the planes as for the diameter AB .

The same property holds in a vacuum; see Art. 67.

297. *A body, acted on by gravity, is projected from a given point with a given velocity and in a given direction in a medium, the resistance of which varies as the velocity. To determine the path of the projectile.*

Take the point of projection for the origin of co-ordinates, the axis of x being parallel, and that of y perpendicular to the horizon.

u = the velocity of projection,

α = the elevation of the direction of projection,

x, y the co-ordinates of the body at the time t , and s the length of the path described,

θ = the inclination of the motion to the horizon at the time t .

The retarding force of the medium $= k d_t s$, which being resolved, gives

$$k d_t s \cos \theta = k d_t s \cdot d_t x = k d_t x$$

parallel to x ; and similarly $k d_t y$ parallel to y ;

$$\therefore d_t^2 x = -k d_t x,$$

$$\text{and } d_t^2 y = -g' - k d_t y.$$

The first equation we have shewn how to integrate in Art. 295, and by proceeding in the same manner, we have

$$x = \frac{u \cos \alpha}{k} (1 - e^{-kt}).$$

The second equation may be put under the form

$$d_t^2 \left(y + \frac{g' t}{k} \right) = -k d_t \left(y + \frac{g' t}{k} \right),$$

which is of the same form as the first, and may be integrated by the same method;

$$\therefore d_t \left(y + \frac{g' t}{k} \right) = C - k \left(y + \frac{g' t}{k} \right).$$

But when $t = 0$, $y = 0$, $d_t y = u \sin \alpha$, and therefore $C = u \sin \alpha + \frac{g'}{k}$;

$$\therefore d_t \left(y + \frac{g' t}{k} \right) = u \sin \alpha + \frac{g'}{k} - k \left(y + \frac{g' t}{k} \right).$$

Dividing this equation by its right-hand member and integrating, we have

$$kt = -\log_e \left\{ u \sin \alpha + \frac{g'}{k} - k \left(y + \frac{g't}{k} \right) \right\} + C.$$

But when $t = 0$, $y = 0$; and $\therefore C = \log_e \left(u \sin \alpha + \frac{g'}{k} \right)$;

$$\begin{aligned} \therefore ky + g't &= \left(u \sin \alpha + \frac{g'}{k} \right) (1 - e^{-kt}) \\ &= \left(u \sin \alpha + \frac{g'}{k} \right) \cdot \frac{kx}{u \cos \alpha}. \end{aligned}$$

$$\text{But } t = \frac{1}{k} \log \frac{u \cos \alpha}{u \cos \alpha - kx};$$

$$\therefore y = (\tan \alpha + \frac{g'}{ku} \sec \alpha) x + \frac{g'}{k^2} \log \left(1 - \frac{kx}{u \cos \alpha} \right),$$

the equation of the path of the projectile.

298. By expanding the expression for y , we find

$$y = x \tan \alpha - \frac{g' \sec^2 \alpha}{2u^2} x^2 - \frac{k g' \sec^3 \alpha}{3u^3} x^3 - \dots$$

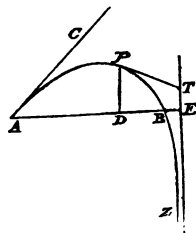
which coincides with the equation in Art. 76 when $k = 0$. The terms containing k shew the effect of the resistance of the medium upon the path of the body.

299. Since the logarithm of a negative quantity is imaginary, and of zero negative and infinite, the

value of x cannot exceed $\frac{u}{k} \cos \alpha$, and when

$x = \frac{u}{k} \cos \alpha$, $y = -\infty$; consequently this

ordinate is an asymptote to the path: it is represented by EZ .



x may increase in a negative direction without limit, but there is no asymptote to that branch of the curve. Also, since

$$d_s y = \tan \alpha + \frac{g'}{ku} \sec \alpha - \frac{g'}{k} \cdot \frac{1}{u \cos \alpha - kx},$$

the inclination of the path to the horizon when $x = -\infty$, is

$$\tan^{-1} \left(\tan \alpha + \frac{g'}{ku} \sec \alpha \right).$$

300. The range AB on the horizontal plane may be found by making $y = 0$ in the equation to the path.

Also by making $d_s y = 0$, we find

$$x = \frac{u^2 \sin \alpha \cos \alpha}{g' + ku \sin \alpha},$$

which therefore gives the greatest altitude y

$$= \frac{u}{k} \sin \alpha - \frac{g'}{k^2} \log_e \left(1 + \frac{ku}{g'} \sin \alpha \right).$$

301. Let the tangent at P be produced to meet the asymptote in T . Then

$$\begin{aligned} PT &= (AE - x) \sec \theta \\ &= \left(\frac{u}{k} \cos \alpha - x \right) d_s s \\ &= \left(\frac{u}{k} \cos \alpha - x \right) \frac{d_t s}{d_t x} = \frac{d_t s}{k}; \end{aligned}$$

$$\therefore d_t x = u \cos \alpha - kx;$$

$$\therefore \text{the velocity} = k \cdot PT \propto PT.$$

Consequently the velocity at any point is proportional to the length of the tangent intercepted between the body and the asymptote.

Also, since

$$ky + g't = \left(u \sin \alpha + \frac{g'}{k} \right) (1 - e^{-kt});$$

$$\therefore k d_t y + g' = (ku \sin \alpha + g') e^{-kt}.$$

Consequently the ultimate value of $d_t y$ is $-\frac{g'}{k}$, which is therefore the ultimate value of the velocity of the body. Hence the ultimate value of PT is $\frac{g'}{k^2}$.

302. *A heavy body descends from rest in a medium, the resistance of which varies as the square of the velocity, to find the space described in a given time.*

Let s be the space descended in the time t ;

$$\therefore d_t^2 s = g' - k (d_t s)^2;$$

$$\therefore \sqrt{k} = \frac{\sqrt{k} d_t^2 s}{g' - k (d_t s)^2};$$

$$\therefore \sqrt{k} \cdot t = \frac{1}{2\sqrt{g'}} \cdot \log_e \frac{\sqrt{g'} + \sqrt{k} d_t s}{\sqrt{g'} - \sqrt{k} d_t s}.$$

No constant is added, because when $t = 0$, $d_t s = 0$;

$$\begin{aligned} \therefore k d_t s &= \sqrt{k g'} \cdot \frac{e^{2\sqrt{k g'} \cdot t} - 1}{e^{2\sqrt{k g'} \cdot t} + 1}; \\ &= \sqrt{k g'} \cdot \frac{e^{\sqrt{k g'} \cdot t} - e^{-\sqrt{k g'} \cdot t}}{e^{\sqrt{k g'} \cdot t} + e^{-\sqrt{k g'} \cdot t}}; \end{aligned}$$

$$\therefore k s = \log_e (e^{\sqrt{k g'} \cdot t} + e^{-\sqrt{k g'} \cdot t}) + C.$$

Now when $t = 0$, $s = 0$, and therefore $C = -\log_e 2$;

$$\therefore k s = \log_e \frac{e^{\sqrt{k g'} \cdot t} + e^{-\sqrt{k g'} \cdot t}}{2}.$$

303. *A body is projected vertically upwards in the air, the resistance of which varies as the square of the velocity; having given the velocity of projection, find the height to which it will ascend, and the velocity with which it will return to the point of projection.*

Let A be the point of projection, B the point to which it will ascend, P the place of the body at the time t from leaving A . B
┌
└ P
A

u = the velocity of projection, u' = the velocity with which it returns to A .

$s = AP$, v = the velocity at P .

Then the equation of motion for the ascent is

$$d_t^2 s = -g' - k(d_t s)^2,$$

or since $d_t s = v$,

$$v d_t v = -g' - k v^2;$$

$$\therefore 2 = -\frac{2v d_t v}{g' + k v^2},$$

$$\text{and } 2s = -\frac{1}{k} \cdot \log_e (g' + k v^2) + C.$$

But when $s = 0$, $v = u$, and $\therefore C = \frac{1}{k} \cdot \log_e (g' + k u^2)$;

$$\therefore 2s = \frac{1}{k} \cdot \log_e \frac{g' + k u^2}{g' + k v^2}.$$

At the point B , $s = AB$ and $v = 0$;

$$\therefore AB = \frac{1}{2k} \cdot \log_e \left(1 + \frac{k}{g'} u^2 \right).$$

For the descent.

The body begins to fall from rest at B , and therefore calling BP , s , the equation of motion is

$$d_t^2 s = g' - k(d_t s)^2,$$

or if v' be the velocity of descent at P ,

$$v' d_t v' = g' - k v'^2;$$

$$\therefore 2 = \frac{2v' d_t v'}{g' - k v'^2},$$

$$\text{and } 2s = -\frac{1}{k} \log_e (g' - kv^2) + C.$$

$$\text{At } B, s = 0 \text{ and } v' = 0; \therefore C = \frac{1}{k} \log_e g';$$

$$\therefore 2s = \frac{1}{k} \log_e \frac{g'}{g' - kv'^2}.$$

$$\text{Now at } A, s = AB, \text{ and } v' = u';$$

$$\therefore AB = \frac{1}{2k} \log_e \frac{g'}{g' - ku'^2};$$

$$\therefore \frac{g'}{g' - ku'^2} = 1 + \frac{k}{g'} u'^2;$$

$$\therefore \frac{1}{u'^2} = \frac{1}{u^2} + \frac{k}{g'};$$

from which u' the velocity of return is known.

304. *A body, acted on by gravity, is projected from a given point with a given velocity and in a given direction, in the air, the resistance of which varies as the square of the velocity. To determine the path of the projectile.*

Let us use the notation of Art. 297, then the equations of motion are

$$d_t^2 x = -k (d_t s)^2 \cos \theta = -k (d_t s)^2 d_x,$$

$$\text{and } d_t^2 y = -g' - k (d_t s)^2 \sin \theta = -g' - k (d_t s)^2 d_y.$$

These equations will be rendered much more simple by changing the independent variable from t to x . By this means they are reducible to

$$\frac{d_x^2 t}{d_x} = k d_x s, \dots\dots\dots(1),$$

$$\text{and } d_x^2 y = -g' (d_x t)^2 \dots\dots\dots(2).$$

The former of which by integration gives

$$\log_e (d_x t) = ks + C.$$

But at the point of projection $s = 0$, and $d_s t = \frac{1}{u \cos \alpha}$;

$$\therefore C = -\log_e (u \cos \alpha),$$

$$\text{and } \log_e (d_s t) + \log_e (u \cos \alpha) = ks;$$

$$\therefore d_s t = \frac{e^{ks}}{u \cos \alpha} \dots\dots\dots (3).$$

Hence by substitution in (2)

$$d_s^2 y = -\frac{g'}{u^2 \cos^2 \alpha} \cdot e^{2ks} \dots\dots\dots (4).$$

Write p for $d_s y$, and multiply both sides of this by $d_s s$ or $\sqrt{1+p^2}$,

$$\sqrt{1+p^2} \cdot d_s p = -\frac{g'}{u^2 \cos^2 \alpha} \cdot e^{2ks} d_s s;$$

consequently, by integrating this,

$$C + p \sqrt{1+p^2} + \log_e (p + \sqrt{1+p^2}) = -\frac{g' (e^{2ks} - 1)}{k u^2 \cos^2 \alpha};$$

$$\therefore \frac{e^{2ks} - 1}{2k} = -\frac{u^2}{2g'} \cos^2 \alpha \{p \sqrt{1+p^2} + \log_e (p + \sqrt{1+p^2}) + C\} \dots (5).$$

Now if S be the length of the arc described by the same projectile in vacuo, comprehended between the point of projection and the point where the inclination of the path is the same as at the extremity of the arc s in the medium, then by Art. 82,

$$\frac{e^{2ks} - 1}{2k} = S;$$

$$\therefore 2ks = \log_e (1 + 2kS) \dots\dots (6).$$

The equation of the curve described by the projectile must be determined by substituting $d_s y$ for p in equation (5), and then integrating. No method, however, of effecting this integration in finite terms has yet been discovered, although the problem before us has exercised the abilities of

the most celebrated writers on Mechanics, in consequence of its great importance in the Science of Gunnery.

305. From equation (6), we learn that s and S increase together, but S much faster than s ; and that they are infinite together. Now we know that when S is infinite, p is also infinite. Consequently, the descending branch of the curve is ultimately vertical. To find the ultimate value of x , let us suppose p very large. Then by differentiating the equation (4), or

$$d_x p = - \frac{g'}{u^2 \cos^2 \alpha} e^{2ks},$$

we find

$$\begin{aligned} d_x^2 p &= - \frac{2g'k}{u^2 \cos^2 \alpha} e^{2ks} \sqrt{1+p^2} \\ &= 2k d_x p \sqrt{1+p^2} \\ &= 2kp d_x p; \because p \text{ is very large;} \\ \therefore d_x p &= kp^2, \end{aligned}$$

no constant being added because it vanishes compared with p^2 .

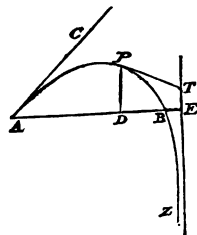
Dividing by p^2 and integrating, we find

$$\frac{1}{p} = C' - kx.$$

Since this equation is true, when p is not infinite but only very large, C' is necessarily finite; and when p is actually infinite,

$$x = \frac{C'}{k}.$$

Consequently, if AE be made equal to $\frac{C'}{k}$, EZ which is perpendicular to AE , will be an asymptote to the descending branch.



306. In the ascending branch s and S are both negative, and equation (6) becomes

$$-2ks = \log(1 - 2kS),$$

which shews that s will be infinite when S has the finite value $\frac{1}{2k}$.

If, therefore, in a negative direction, we take an arc of the common parabola, which is the projectile in vacuo, equal to $\frac{1}{2k}$, the tangent at its extremity will be parallel to the ascending branch at an infinite distance. This proves that the ascending and descending branches are dissimilar, the body rising obliquely and ultimately descending vertically.

From equation (5) it appears, that when s has a very large negative value, p is very nearly constant;

$$\therefore d_s y = A \text{ nearly;}$$

$$\therefore y = Ax + A',$$

which being true when x and y are not infinite, but only very large, A' must be finite; and consequently the ascending branch has an asymptote whose equation is

$$y = Ax + A'.$$

307. In the practice of gunnery, p is always small, and consequently, we may neglect terms which contain higher powers of p than p^2 . Hence,

$$\begin{aligned} -\frac{g'}{u^2 \cos^2 \alpha} \cdot e^{2s} d_s s &= \sqrt{1 + p^2} \cdot d_s p, \\ &= (1 + \frac{1}{2} p^2) d_s p; \end{aligned}$$

$$\therefore C - \frac{g' (e^{2s} - 1)}{2k u^2 \cos^2 \alpha} = p + \frac{1}{6} p^3 = p.$$

But when $s = 0$, $p = \tan \alpha$; $\therefore C = \tan \alpha$;

$$\therefore e^{2s} = 1 + \frac{2k u^2}{g'} \cos^2 \alpha (\tan \alpha - p).$$

Substituting this in equation (4)

$$d_s p (= d_s^2 y) = -\frac{g'}{u^2 \cos^2 \alpha} - 2k (\tan \alpha - p);$$

$$\therefore d_x p - 2kp = - \left(2k \tan \alpha + \frac{g'}{u^2 \cos^2 \alpha} \right).$$

Multiply by e^{2kx} and integrate;

$$\therefore p e^{-2kx} = C + \left(\tan \alpha + \frac{g' \sec^2 \alpha}{2ku^2} \right) e^{-2kx}.$$

But when $x = 0$, $p = \tan \alpha$; $\therefore C = - \frac{g' \sec^2 \alpha}{2ku^2}$;

$$\therefore d_x y (= p) = \tan \alpha - \frac{g' \sec^2 \alpha}{2ku^2} (e^{2kx} - 1);$$

$$\therefore y = x \left(\tan \alpha + \frac{g' \sec^2 \alpha}{2ku^2} \right) - \frac{g' e^{2kx}}{4k^2 u^2 \cos^2 \alpha} + C.$$

But when $x = 0$, $y = 0$; and $\therefore C = \frac{g'}{4k^2 u^2 \cos^2 \alpha}$;

$$\therefore y = x \left(\tan \alpha + \frac{g' \sec^2 \alpha}{2ku^2} \right) - \frac{g' (e^{2kx} - 1)}{4k^2 u^2 \cos^2 \alpha},$$

which is the equation of the projectile.

308. *A body moving in a resisting medium is acted on by a force in parallel lines; to find the resistance that a given curve may be described.*

Let the axis of y be parallel to the direction in which the force acts, and represent the force acting at any point xy by Y and the resistance by R . Then the equations of motion are

$$d_t^2 x = -R d_x \dots \dots (1),$$

$$\text{and } d_t^2 y = Y - R d_y.$$

Eliminating R we have

$$d_t x d_t^2 y - d_t y d_t^2 x = Y d_t x;$$

$$\therefore d_t^2 y = \frac{d_t x d_t^2 y - d_t y d_t^2 x}{(d_t x)^2} = \frac{Y}{(d_t x)^2};$$

$$\therefore (d_t x)_\perp = \frac{Y}{d_t^2 y},$$

$$\therefore 2 d_t x d_t^2 x = d_s \left(\frac{Y}{d_s^2 y} \right) d_t x;$$

$$\therefore d_t^2 x = \frac{1}{2} d_s \left(\frac{Y}{d_s^2 y} \right);$$

and substituting this result in equation (1), we have

$$R = -\frac{1}{2} d_s s d_s \left(\frac{Y}{d_s^2 y} \right).$$

309. *If the resistance vary as the product of the density into the square of the velocity, to find the density, that a given curve may be described; the force acting as before in parallel lines.*

Let $R = Qv^2 = Q(d_t s)^2$; then the equations of motion are

$$d_t^2 x = -Q(d_t s)^2 d_t x = -Q d_t s d_t x \dots \dots (1),$$

$$d_t^2 y = Y - Q(d_t s)^2 d_t y = Y - Q d_t s d_t y.$$

Eliminating Q between these equations, we find

$$d_t x d_t^2 y - d_t y d_t^2 x = Y d_t x;$$

$$\therefore d_s^2 y = \frac{d_t x d_t^2 y - d_t y d_t^2 x}{(d_t x)^3} = \frac{Y}{(d_t x)^2};$$

$$\therefore \log_e \frac{Y}{d_s^2 y} = 2 \log_e d_t x,$$

and by differentiating this equation

$$d_t \log_e \frac{Y}{d_s^2 y} = \frac{2 d_t^2 x}{d_t x} = -2 Q d_t s, \text{ from (1);}$$

$$\therefore Q = \frac{1}{2} d_s \log_s \frac{d_s^2 y}{Y}.$$

310. *On the same supposition, if the density be given, we may find the force by which a given curve may be described.*

For from the last found equation

$$\log_e \left(\frac{d^2 y}{Y} \right) = 2 \int Q + \text{constant};$$

$$\therefore Y = C e^{-2 \int Q} d^2 y.$$

311. *The velocity at any point is that which is due to one fourth of the chord of curvature at that point.*

The equations of motion are

$$d_t^2 x = -R d_s x = -R \frac{d_x x}{d_s},$$

$$d_t^2 y = Y - R d_s y = Y - R \frac{d_y y}{d_s}.$$

Eliminating R , we get

$$Y d_s x = d_s x d_t^2 y - d_s y d_t^2 x;$$

$$\therefore \frac{(d_s)^2}{Y d_s x} = \frac{(d_s)^2}{d_s x d_t^2 y - d_s y d_t^2 x} = \text{rad. curv. } (\rho);$$

$$\begin{aligned} \therefore (d_s)^2 &= Y \cdot \rho \frac{d_s x}{d_s} = Y \cdot \rho d_s \\ &= 2Y \cdot \frac{1}{4} (\text{chd. curv.}). \end{aligned}$$

Consequently the space due to the velocity is equal to one fourth of the chord of curvature.

312. *A body moving in a resisting medium is acted on by a central force; to find the resistance that a given curve may be described.*

Let us use the notation of Art. 107. The equations of motion are

$$d_t^2 x = -F \frac{x}{r} - R d_s x = -F \frac{x}{r} - R \frac{d_x x}{d_s},$$

$$\text{and } d_t^2 y = -F \frac{y}{r} - R d_s y = -F \frac{y}{r} - R \frac{d_y y}{d_s}.$$

Multiply these by $d_t y$ and $d_t x$ respectively, and subtract;

$$\therefore d_t x d_t^2 y - d_t y d_t^2 x = \frac{F}{r} (x d_t y - y d_t x);$$

$$\text{or, } \frac{(d_t s)^3}{\rho} = \frac{F}{r} \cdot p d_t s \text{ from Diff. Calc.}$$

$$\begin{aligned} \therefore (d_t s)^2 &= F \cdot \frac{\rho p}{r} = F \cdot r d_p r \cdot \frac{p}{r} \\ &= F p d_p r \dots\dots(1). \end{aligned}$$

Again, eliminating F from the equations of motion above given, we have

$$\begin{aligned} x d_t^2 y - y d_t^2 x &= -\frac{R}{d_t s} (x d_t y - y d_t x) \\ &= -\frac{R}{d_t s} \cdot p d_t s \\ &= -R p. \end{aligned}$$

$$\begin{aligned} \text{But } x d_t^2 y - y d_t^2 x &= d_t (x d_t y - y d_t x) \\ &= d_t (p d_t s); \end{aligned}$$

$$\therefore -R p = d_t (p d_t s);$$

$$\begin{aligned} \therefore -2 R p^2 &= 2 (p d_t s) d_t (p d_t s) \\ &= d_t (p d_t s)^2 \\ &= d_t (F p^3 d_p r) \text{ from (1);} \end{aligned}$$

$$\therefore R = -\frac{d_t (F p^3 d_p r)}{2 p^2}.$$

313. If the resistance varies as the product of the density into the square of the velocity,

$$\begin{aligned} \text{let } R &= Q v^2 \\ &= Q F p d_p r, \text{ from (1);} \\ \therefore Q &= -\frac{d_t (F p^3 d_p r)}{2 F p^3 d_p r} \\ &= -\frac{1}{2} d_t \log_e (F p^3 d_p r). \end{aligned}$$

If F' be the force by which the curve may be described in vacuo, then

$$F' = \frac{h^2 d_r p}{p^3} \dots\dots \text{Art. (118)};$$

$$\therefore p^3 d_r r = \frac{h^2}{F'};$$

$$\therefore Q = -\frac{1}{2} d_r \log_e \left(\frac{F'}{F} \right)$$

$$= \frac{1}{2} d_r \log_e \left(\frac{F'}{F} \right).$$

314. *Having given the density, to find the central force by which a given curve may be described; the resistance varying as the product of the density into the square of the velocity.*

We have shewn that

$$Q = -\frac{1}{2} d_r \log_e (F p^3 d_r r);$$

$$\therefore 2 \int_r Q + \text{const.} = -\log_e (F p^3 d_r r),$$

$$\text{whence } F p^3 d_r r = C e^{-2 \int_r Q};$$

$$\text{and } \therefore F = C e^{-2 \int_r Q} \cdot \frac{d_r p}{p^3}$$

$$= F' \cdot e^{-2 \int_r Q} \dots \text{Art. (118).}$$

315. *On the same supposition, having given the density and force, to find the orbit.*

The equations of motion are

$$d_t^2 x = -F \frac{x}{r} - R \frac{d_t x}{d_t s} = -F \frac{x}{r} - Q d_t s d_t x,$$

$$d_t^2 y = -F \frac{y}{r} - R \frac{d_t y}{d_t s} = -F \frac{y}{r} - Q d_t s d_t y;$$

$$\therefore x d_t^2 y - y d_t^2 x = -Q d_t s (x d_t y - y d_t x).$$

Dividing by $x d_t y - y d_t x$, and integrating, we have

$$\log_e (x d_t y - y d_t x) = - \int Q + \text{const.};$$

$$\therefore x d_t y - y d_t x = h e^{-\int Q},$$

$$\text{or } p d_t s = h e^{-\int Q}.$$

Again, eliminating Q from the equations of motion, we find

$$d_t x d_t^2 y - d_t y d_t^2 x = \frac{F}{r} (x d_t y - y d_t x),$$

$$\text{or } \frac{(d_t s)^3}{\rho} = \frac{F}{r} p d_t s;$$

$$\therefore (d_t s)^2 = F \cdot \frac{\rho p}{r} = F p d_p r;$$

$$\therefore F p^3 d_p r = p^2 (d_t s)^2 \\ = h^2 e^{-2\int Q}.$$

$$\text{or } F = \frac{h^2 d_p}{p^3} \cdot e^{-2\int Q}.$$

$$\text{Now } \frac{1}{p^2} = u^2 + (d_\theta u)^2;$$

$$\therefore -\frac{d_\theta p}{p^3} = (u + d_\theta^2 u) d_\theta u;$$

$$\therefore -\frac{d_r p}{p^3} = (u + d_\theta^2 u) \frac{d_\theta u}{d_\theta r}$$

$$= -u^2 (u + d_\theta^2 u), \therefore r = \frac{1}{u};$$

$$\therefore F = h^2 u^2 (u + d_\theta^2 u) e^{-2\int Q}.$$

316. *To find the time of oscillation in the arc of a cycloid, whose axis is vertical and base horizontal, the motion taking place in a uniform medium, the resistance of which varies as the velocity. See Art. 156.*

Here the resistance = kv ; and employing the usual notation for the cycloid,

$$d_t^2 s = -\frac{s}{l} \cdot g' + kv.$$

But in the descent $v = -d_t s$;

$$\therefore d_t^2 s + k d_t s + \frac{g'}{l} s = 0 \dots (1).$$

When the body ascends through the other arc of the cycloid after passing the lowest point, if s' be the arc ascended the equation of motion for the ascent is

$$d_t^2 s' = -\frac{s'}{l} \cdot g' - k v.$$

But $v = d_t s'$, and $s' = -s$, therefore this equation becomes

$$-d_t^2 s = +\frac{s}{l} \cdot g' + k d_t s,$$

which is the same as before. Hence equation (1) is true both for the ascent and descent. Since it is a linear equation of the second order, its integral is

$$s = e^{-ikt} (A \sin ht + B \cos ht) \dots (2),$$

h being such a quantity that $-\frac{1}{2}k \pm h\sqrt{-1}$ are the two roots of the equation

$$m^2 + km + \frac{g'}{l} = 0;$$

$$\text{and } \therefore h^2 = \frac{g'}{l} - \frac{k^2}{4}.$$

The roots are imaginary, because k is supposed to be less than $2\left(\frac{g'}{l}\right)^{\frac{1}{2}}$.

In order to determine the values of the arbitrary constants A, B , let us differentiate equation (2);

$$\begin{aligned} \therefore d_t s &= e^{-ikt} (hA \cos ht - hB \sin ht) \\ &\quad - \frac{1}{2}ke^{-ikt} (A \sin ht + B \cos ht). \end{aligned}$$

But when $t = 0$, $d_t s = 0$;

$$\therefore 0 = hA - \frac{1}{2}kB.$$

And if β be the length of the arc of descent, then when $t = 0$, $s = \beta$ which being written in equation (2), gives

$$\beta = B;$$

$$\therefore A = \frac{k\beta}{2h};$$

$$\therefore s = \frac{\beta}{2h} (k \sin ht + 2h \cos ht) e^{-ikt} \dots\dots\dots (3),$$

$$\begin{aligned} \text{and } d_t s &= - \left(h + \frac{k^2}{4h} \right) \beta e^{-ikt} \sin ht \\ &= - \frac{g'\beta}{hl} \cdot e^{-ikt} \sin ht \dots\dots\dots (4). \end{aligned}$$

Now at the beginning and end of each oscillation $d_t s = 0$, and therefore $\sin ht = 0$;

$$\therefore ht = 0, \pi, 2\pi, 3\pi, \dots\dots$$

and therefore the time of an oscillation

$$(\text{= } T) = \frac{\pi}{h} = \frac{\pi}{\left(\frac{g'}{l} - \frac{k^2}{4} \right)^{\frac{1}{2}}}.$$

This being independent of the magnitude of the arc of oscillation, shews that the cycloid is tautochronous in this medium as well as in a vacuum.

317. The velocity will be a maximum when

$$e^{-ikt} \sin ht \text{ is a maximum,}$$

$$\text{or when } 0 = e^{-ikt} \left(h \cos ht - \frac{1}{2} k \sin ht \right);$$

$$\therefore \tan ht = \frac{2h}{k};$$

$$\therefore \sin ht = h \left(\frac{l}{g'} \right)^{\frac{1}{2}} \text{ and } \cos ht = \frac{k}{2} \left(\frac{l}{g'} \right)^{\frac{1}{2}};$$

$$\text{and } \therefore \text{ when } s = \beta k \left(\frac{l}{g'} \right)^{\frac{1}{2}} e^{-\frac{k}{2h} \tan^{-1} \left(\frac{2h}{k} \right)}.$$

318. At the lowest point $s = 0$, and $k \sin ht + 2h \cos ht = 0$;

$$\therefore \tan ht = -\frac{2h}{k},$$

$$\text{or } \tan(\pi - ht) = \frac{2h}{k}.$$

Hence if t' be the time when the velocity is a maximum, and t'' the time of passing the lowest point,

$$\pi - ht'' = ht';$$

$$\therefore t' + t'' = \frac{\pi}{h}$$

= time of an oscillation

$$= T;$$

$$\therefore t' = T - t''.$$

But $T - t''$ is the time of ascent; consequently, the time of ascent is equal to that part of the time of the descent which is spent in acquiring the maximum velocity.

The velocity at the lowest point

$$= \frac{g'\beta}{hl} e^{-ikt''} \sin ht''.$$

But $\sin ht'' = \sin ht'$, and therefore

$$\frac{\text{vel. at lowest point}}{\text{maximum vel.}} = \frac{e^{ikt'}}{e^{ikt''}} = e^{-ik(t''-t')}.$$

319. By writing π for ht in (3), we have the arc of ascent

$$= \beta e^{-\frac{h\pi}{2k}};$$

$$\therefore \frac{\text{an arc of descent}}{\text{next arc of ascent}} = e^{\frac{h\pi}{2k}}.$$

This being a constant ratio, the successive arcs are in geometrical progression. Hence the whole space described before the motion ceases

$$\begin{aligned}
 &= \beta + 2\beta e^{-\frac{k\pi}{2h}} + 2\beta e^{-\frac{2k\pi}{2h}} + \dots \\
 &= \beta \cdot \frac{e^{\frac{k\pi}{2h}} + 1}{e^{\frac{k\pi}{2h}} - 1}.
 \end{aligned}$$

320. To determine the same as in Art. 316, the resistance of the medium being supposed to vary as the square of the velocity.

In this case the retarding force of the medium $= kv^2$, and therefore the equation of motion for the descent, is

$$d_t^2 s = -\frac{g}{l} \cdot s + kv^2.$$

$$\text{But } -d_t s = v;$$

$$\therefore -d_t^2 s = d_t v = d_t v \cdot d_t s = -v d_t v;$$

$$\therefore v d_t v - kv^2 = -\frac{g}{l} \cdot s.$$

The same equation will be found to be true for the ascent. Multiply by $2e^{-2ks}$ and integrate;

$$\begin{aligned}
 \therefore v^2 e^{-2ks} &= -\frac{2g'}{l} \int_s^{-2ks} s \\
 &= \frac{(2ks + 1)g'}{2k^2 l} \cdot e^{-2ks} + C.
 \end{aligned}$$

But when $s = \beta$, $v = 0$, and

$$0 = \frac{(2k\beta + 1)g'}{2k^2 l} \cdot e^{-2k\beta} + C;$$

$$\therefore v^2 = \frac{(2ks + 1)g'}{2k^2 l} - \frac{(2k\beta + 1)g'}{2k^2 l} e^{-2k(\beta - s)}.$$

It does not appear that this equation admits of integration in finite terms; we may, however, obtain approximate results when the density of the medium is small, by expanding $e^{-2k(\beta - s)}$ as far as terms containing k^2 .

By this means we find

$$\left(\frac{g'}{l}\right)^{\frac{1}{2}} = \frac{-d_t s}{\sqrt{\beta^2 - s^2}} \cdot \left\{1 + \frac{k}{3} \left(\frac{\beta - s}{\beta + s}\right) (2\beta + s) + \dots\right\};$$

$$\therefore \left(\frac{g'}{l}\right)^{\frac{1}{2}} \cdot t' = \cos^{-1} \frac{s}{\beta} + \frac{k}{3} \frac{(\beta - s)^{\frac{3}{2}}}{(\beta + s)^{\frac{3}{2}}}.$$

No constant is added, because $t = 0$, when $s = \beta$. When $s = 0$, let $t = t'$;

$$\therefore \left(\frac{g'}{l}\right)^{\frac{1}{2}} \cdot t' = \frac{\pi}{2} + \frac{k\beta}{3}.$$

Consequently the excess of the time of descent above that in vacuo, omitting the effect of buoyancy, is

$$= \frac{k\beta}{3} \cdot \left(\frac{l}{g'}\right)^{\frac{1}{2}}.$$

321. *To determine the motion of a body acted on by gravity, and descending in a circular arc in a uniform medium, the resistance of which varies as the square of the velocity.*

Let a = the radius of the circle, then the force of gravity down the arc $= g' \sin \frac{s}{a}$, and the retarding force of the medium $= kv^2$, consequently the equation of motion is

$$v dv = -g' \sin \frac{s}{a} + kv^2,$$

$$\text{or } v dv - kv^2 = -g' \sin \frac{s}{a},$$

which is true both for the descent and ascent.

Multiply by $2e^{-2ks}$ and integrate, then

$$(1 + 4k^2 a^2) v^2 = 2ag' \left(\cos \frac{s}{a} + 2ak \sin \frac{s}{a} \right) - 2ag' \left(\cos \frac{\beta}{a} + 2ak \frac{\beta}{a} \right) e^{2k(\beta-s)},$$

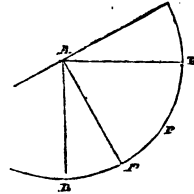
β being the value of s when the motion commences.

322. If the value of β be such that

$$\cos \frac{\beta}{a} + 2ka \sin \frac{\beta}{a} = 0;$$

$$\text{or } 2ka = \tan \left(\frac{\beta}{a} - \frac{\pi}{2} \right),$$

$$\text{then } v^2 = 2ag' \sin \frac{\beta}{a} \sin \left(\frac{\beta}{a} \sin \frac{s}{a} \right).$$



Now if the angle $BAC = \frac{\beta}{a}$, and a body begins to descend in a vacuum from C , acted on by an accelerating force $g' \sin \frac{\beta}{a}$ in the direction AF , then at any point P ,

$$\begin{aligned} \text{its (vel.)}^2 &= 2g' \sin \frac{\beta}{a} \cdot a \sin PAC \\ &= 2ag' \sin \frac{\beta}{a} \sin \left(\frac{\beta}{a} - \frac{s}{a} \right) \\ &= (\text{vel.})^2 \text{ in the resisting medium.} \end{aligned}$$

Consequently the motion of such a body in vacuo will be the same as that of the body in the medium.

323. The general equation of an ellipse referred to its focus as origin being $r = \frac{a(1-e^2)}{1+e \cos(\theta-\varpi)}$, in which there are three disposable constants a, e, ϖ , it can be made to pass through three given points, while it retains a fixed focus. Now the tangent of a curve depends only on *two* consecutive points, and the curvature depends only on *three* consecutive points; consequently we can always find an ellipse having a fixed focus which shall have the same tangent and curvature at a proposed point as a given curve. Hence whatever be the nature of the orbit which a body describes round a fixed centre of force, we can at any time suppose it moving in an ellipse which has a common tangent with the actual orbit, and the same curvature. This may be expressed by saying

that the orbit is an ellipse, the elements (viz. the axis major, eccentricity and position of the axis) of which are continually varying.

324. *PROB. A body moves in a resisting medium acted on by a central force $\propto D^{-2}$; the resistance $\propto$ (density of medium) $\cdot$ (vel.)². To find the effect of the resistance upon the elements of the varying elliptic orbit in which it may be supposed to be moving.*

Let us use the notation of Art. 315, then the equation of the elliptic orbit at the time t is

$$u = \frac{1 + e \cos (\theta - \varpi)}{a (1 - e^2)}$$

which, for convenience, we may put under the form

$$u = A + B \cos (\theta - \varpi) \dots \dots \dots (1).$$

Now because the tangent and curvature of the elliptic orbit are the same as in the actual orbit, therefore u , θ , $d_\theta u$, $d_\theta^2 u$ are likewise the same for both.

$$\text{Hence} \quad d_\theta u = -B \sin (\theta - \varpi) \dots \dots \dots (2),$$

$$\text{and} \quad d_\theta^2 u = -B \cos (\theta - \varpi) \dots \dots \dots (3).$$

These being written in the equation of the actual orbit (Art. 315), viz.

$$F = h^2 u^2 (d_\theta^2 u + u) e^{-2f.Q},$$

$$\text{give} \quad \mu u^2 = h^2 u^2 A e^{-2f.Q},$$

$$\text{or} \quad A = \frac{\mu}{h^2} e^{2f.Q};$$

$$\therefore \text{ lat. rect.} = \frac{2}{A} = \frac{2h^2}{\mu} e^{-2f.Q} \dots \dots \dots (4).$$

This equation shews that the latus rectum of the orbit is continually diminishing, a result which also appears from equation (2) of Art. 129.

Again, by the nature of our hypothesis respecting the elliptic orbit, A , B , ϖ change with θ , yet so as that they always satisfy the equations (1) (2) (3). Hence differentiating (1) and (2) taking account in each case of (2) and (3) respectively, we have

$$0 = d_{\theta} A + d_{\theta} B \cdot \cos(\theta - \varpi) + d_{\theta} \varpi \cdot B \sin(\theta - \varpi),$$

and $0 = d_{\theta} B \cdot \sin(\theta - \varpi) - d_{\theta} \varpi \cdot B \cos(\theta - \varpi).$

From which we find

$$d_{\theta} B = -d_{\theta} A \cdot \cos(\theta - \varpi),$$

$$\text{and } B d_{\theta} \varpi = -d_{\theta} A \cdot \sin(\theta - \varpi):$$

or changing the variable from θ to s , and substituting for $d_{\theta} A$,

$$d_s B = -\frac{2\mu Q}{h^2} e^{2f \cdot Q} \cos(\theta - \varpi) = -2A Q \cos(\theta - \varpi),$$

$$B d_s \varpi = -\frac{2\mu Q}{h^2} e^{2f \cdot Q} \sin(\theta - \varpi) = -2A Q \sin(\theta - \varpi).$$

But $B = Ae$, and $\therefore d_s B = e d_s A + A d_s e = A(2eQ + d_s e)$;

consequently by substitution the last equations become

$$d_s e = -2Q \{e + \cos(\theta - \varpi)\} \dots\dots\dots(5),$$

$$\text{and } d_s \varpi = -\frac{2Q}{e} \sin(\theta - \varpi) \dots\dots\dots(6).$$

Equation (5) shews that the eccentricity *diminishes* as long as the body's distance from the centre of force is *less* than the semi-major axis; and increases in the contrary case.

But the greatest value of the rate ($d_s e$) of diminution is $2Q(e + 1)$, and the greatest value of the rate of increase is $2Q(1 - e)$; and the whole length of arc (s) described in both cases is nearly the same (the difference being caused chiefly by the variation of ϖ); consequently upon the whole the eccentricity is diminished at each revolution. Now in (4) we have seen that $a(1 - e^2)$ diminishes continually; and as the factor $1 - e^2$ increases, it follows that $2a$, the major axis, rapidly diminishes at each revolution. Also since both a ,

and $a(1 - e^2)$ diminish, $\therefore a^2(1 - e^2)$, and consequently the minor axis diminishes rapidly, though less rapidly than the major axis. The effect on the eccentricity may be reversed if the density of the medium increase rapidly from the centre of force, for then the value of $2Q(1 - e)$ at the farther apse may exceed that of $2Q(1 + e)$ at the nearer apse.

The discussion of (6) is not so simple as that of (5), on account of e occurring in the denominator of the expression for d, ϖ , the effect of which will be very considerable when e is small. If e were constant the apsidal line would regrede or progrede according as the body is receding from or approaching towards the centre of force: but as the body approaches *more* than it recedes (for we have shewn that the size of the orbit decreases at each revolution) the apsidal line would upon the whole seem to progress. But this effect will be diminished if the arc (5) described in descending towards the nearer apse be not as large as the arc previously described in ascending towards the farther apse. Upon the whole it is probable the progression of the apse will not be very great, though the oscillation of the major axis on each side of its mean position will be very considerable.

The reader may refer back to Art. 131, where he will find most of these results anticipated.

APPENDIX.

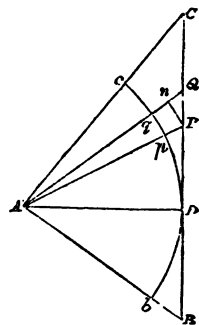
ATTRACTION.

1. KEPLER first discovered, with much labour of observation and calculation, that the Planets describe about the Sun areas proportional to the times of their descriptions; that their orbits are ellipses having the Sun in one of the foci; and that the squares of their periods are proportional to the cubes of the major axes. The first of these laws, compared with *Dynamics*, Art. 122, shews that the force which acts on each planet is directed towards the centre of the Sun; the second, compared with Art. 120, shews that for the same planet this force varies inversely as the square of the distance between the centres of the Sun and planet; from the third, compared with Art. 133, we infer that it is the same force which acts upon all the planets, so that if they were all removed to the same distance from the Sun's centre, they would fall towards it with the same velocity; and, therefore, the respective gravitating forces of the planets are proportional to their masses divided by the squares of their distances from his centre. The Laws of Kepler are found to be equally applicable to the motion of satellites about their primaries; and therefore such of the planets as have satellites, exert attractive influences which are directed towards their centres, and vary in intensity in the inverse ratio of the squares of the distances from their centres. Analogy leads us to conclude that the other planets exert similar influences, varying according to the same law. Sir Isaac Newton proved that the Moon is retained in her orbit by the same force as that which causes the descent of bodies near the Earth's surface. It is therefore probable, that every particle of matter is endued with the power of attracting every

other particle, and that the attraction of the Sun and planets is the resultant of the attractions of their several particles. It will be shewn hereafter, that if this be true, each particle must exert an attractive influence, the intensity of which varies according to the same law as that exerted by the Sun and planets. Every molecule, therefore, in the solar system, attracts every other with a force varying as *the number of its particles* (that is, *its mass*) *directly*, and the *square of their distance inversely*: and this is the law of attraction, which, unless it be otherwise stated, will always be supposed to be observed by small elements of matter. On the subject of this article the reader may consult the third book of Newton's *Principia*; chap. i. liv. ii. of the *Mécanique Céleste*; and the *Exposition du Système du Monde* of Laplace; where he will find the arguments stated at considerable length.

2. *To find the attraction of a thin prism, of uniform density, on a given particle.*

Let A be the attracted particle, BC the given straight rod. Draw AD perpendicular to BC , and let $AD = f$, $DP = y$, $PQ = \delta y$, ρ = the density of the rod, S = the area of a section of it, which is supposed to be indefinitely small; F , G the resolved parts of the attraction of PD in the directions AD , DC .



Then the mass of $PQ = \rho S \delta y$, and $AP^2 = f^2 + y^2$; therefore the whole attraction of PQ in the direction AP is ultimately equal to

$$\frac{\rho S \delta y}{f^2 + y^2},$$

and therefore δF , that part of it which acts in the direction AD ,

$$= \frac{\rho S \delta y}{f^2 + y^2} \cdot \frac{AD}{AP}$$

$$= \frac{\rho S f \delta y}{(f^2 + y^2)^{\frac{3}{2}}};$$

$$\therefore d_y F = \frac{\rho S f}{(f^2 + y^2)^{\frac{3}{2}}}.$$

$$\text{Similarly, } \delta G = \frac{\rho S \delta y}{f^2 + y^2} \cdot \frac{PD}{AP}$$

$$= \frac{\rho S y \delta y}{(f^2 + y^2)^{\frac{3}{2}}};$$

$$\therefore d_y G = \frac{\rho S y}{(f^2 + y^2)^{\frac{3}{2}}}.$$

Hence, by integration,

$$F = \frac{\rho S y}{f \sqrt{f^2 + y^2}} + C = \frac{\rho S}{f} \sin DAP + C,$$

$$G = \frac{-\rho S}{\sqrt{f^2 + y^2}} + C' = -\frac{\rho S}{AP} + C';$$

and therefore, for the whole rod,

$$F = \frac{\rho S}{f} \cdot (\sin DAC + \sin DAB);$$

$$G = \rho S \cdot \left(\frac{1}{AB} - \frac{1}{AC} \right).$$

3. With the centre A , and radius AD , describe an arc of a circle, cutting the lines AB , AC , AP , AQ , in b , c , p , q ; and suppose bDc a material arc of the same density and thickness as the rod BC .

Then the attraction of PQ in the direction AP

$$= \frac{\rho S \cdot PQ}{AP^2}.$$

But drawing Pn perpendicular to AQ , we have by the similar triangles APD , PQn ,

$$\frac{PQ}{AP} = \frac{Pn}{AD};$$

and by the similar triangles Apq , APn , we have

$$\frac{Pn}{AP} = \frac{pq}{AD},$$

$$\text{and } \therefore \frac{PQ}{AP^2} = \frac{Pn}{AD \cdot AP} = \frac{pq}{AD^2};$$

therefore the attraction of PQ in the direction AP

$$= \frac{\rho S \cdot pq}{AD^2},$$

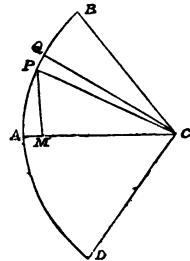
which is equal to the attraction of the circular arc pq .

Hence the particle A is attracted by the straight line BC in the same manner as by the circular arc bc ; and therefore A is attracted in a direction bisecting the angle BAC .

4. PROB. To find the attraction of a uniform circular arc on a particle in the centre of the circle.

Let C be the centre, BAD the arc, A its middle point, join CA , that is the line in which the resultant attraction acts. Let PQ be a small element of the arc. Put $BCA = \alpha$, $PCA = \theta$, $QCP = \delta\theta$, $AC = a$, μ = mass of a unit of length of the arc; then the mass of $PQ = \mu a \delta\theta$, and its attractive force on $C = \frac{\mu a \delta\theta}{a^2}$, and consequently the effective attraction of PQ upon C

$$= \frac{\mu \delta\theta}{a} \cos \theta.$$



This being integrated, and the integral taken between

the limits $-a$, $+a$ of θ , we have the attraction of the whole arc

$$= \frac{2\mu}{a} \sin a.$$

5. PROB. *Two straight rods, situated in the same straight line, are mutually attracted towards each other; to determine the force necessary to keep them asunder.*

By Art. 2, the attraction of the rod AB whose density $= \rho$, and the area of a section of which is S , $= \rho S \left(\frac{1}{BP} - \frac{1}{AP} \right)$ on a unit at P ; also the mass of an element PQ is $\rho' S' \delta x$, and therefore the weight of the element PQ

$$= \rho S \left(\frac{1}{BP} - \frac{1}{AP} \right) \cdot \rho' S' \delta x \dots \text{Art. 23, Dynamics}$$

$$= \rho \rho' S S' \left(\frac{\delta x}{x} - \frac{\delta x}{AB + x} \right);$$

where ρ' , S' are the density and area of a section of the rod CD , $x = BP$, and $\delta x = PQ$;

$$\therefore \text{weight of } CP = \rho \rho' S S' \int_x \left(\frac{1}{x} - \frac{1}{AB + x} \right)$$

$$= \rho \rho' S S' \cdot \log_e \frac{x}{AB + x} + C;$$

$$\therefore \text{weight of } CD = \rho \rho' S S' \cdot \log_e \left(\frac{BD \cdot AC}{BC \cdot AD} \right).$$

Similarly, weight of $AB = \rho \rho' S S' \cdot \log_e \left(\frac{BD \cdot AC}{BC \cdot AD} \right)$, by

changing B into D , and C into A , and conversely: which is equal to the weight of CD , as it ought to be, from Art. 29, Dynamics.

6. *To find the attraction of a circular plate, of uniform density and of inconsiderable thickness, on a particle situated in a line through its centre perpendicular to its plane.*

Let h be the thickness of the plate, f the distance of the attracted particle from its centre. Then the mass of a concentric circular annulus, whose radii are r , $r + \delta r$, is ultimately equal to $2\pi\rho h r \delta r$, and the attraction (δF) of this on the given particle in direction of f

$$= \frac{2\pi\rho h f r \delta r}{(f^2 + r^2)^{\frac{3}{2}}},$$

we estimate the attraction of the element in *this* direction, because the attraction of the whole plate evidently is in this direction; and, therefore, the attractions of the elementary portions of the plate in directions at right angles to this destroy each other;

$$\begin{aligned} \therefore F &= \int_r \frac{2\pi\rho h f r}{(f^2 + r^2)^{\frac{3}{2}}} \\ &= 2\pi\rho h \cdot \left(1 - \frac{f}{\sqrt{f^2 + r^2}}\right) \end{aligned}$$

7. The attraction of all circular plates, of the same thickness and density, on a particle situated as in the last article, will be the same, if their apparent magnitudes, as seen from the attracted point, be the same.

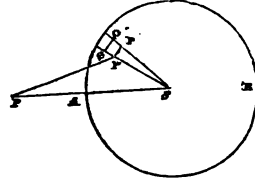
8. When r is infinite, we have the attraction of a plate of unlimited extent $= 2\pi\rho h$, which is independent of the distance of the particle from it.

9. *To determine the attraction of a cone, of uniform density, on a particle at its vertex.*

Let 2θ be the vertical angle of the cone. Then (Art. 6) the attraction of a slice whose thickness is δf parallel to the base, at the distance f from the vertex, is ultimately equal to $2\pi\rho\delta f \cdot (1 - \cos \theta)$; and therefore the attraction of the cone $= 2\pi\rho(1 - \cos \theta) \cdot (\text{altitude})$.

10. *To determine the attraction of a spherical shell of uniform density on a given particle; the attraction of the particles, of which the shell is composed, varying as any function of the distance.*

Let S be the centre of the shell, p the attracted particle, AB the section of the external surface by the plane of the paper; in this plane draw SP, SP' making angles $\theta, \theta + \delta\theta$ with pS ; and let SP, SP' be produced to Q, Q' ; let a, a' be the external and internal radii of the shell,



$$b = Sp, \quad u = pP, \quad r = SP = SP',$$

$\delta r = PQ = P'Q'$, and suppose $\mu\phi u$ = the attraction of a mass μ at the distance u , and let ρ be the density of the shell.

$$\text{Then } u^2 = b^2 - 2br \cos \theta + r^2 \dots\dots(1),$$

$$\text{and } PQ' \text{ ultimately} = r\delta\theta\delta r,$$

δr and $\delta\theta$ being independent increments. Also the mass of the annulus generated by the revolution of this element round Sp

$$\begin{aligned} &= 2\pi\rho r^2\delta r \cdot \sin \theta \cdot \delta\theta \\ &= -2\pi\rho r^2\delta r \cdot \delta(\cos \theta) \\ &= \frac{2\pi\rho r\delta r \cdot u\delta u}{b}, \end{aligned}$$

by eliminating $\cos \theta$ by means of (1), for $u\delta u = -br\delta \cos \theta$, because r and θ are independent.

And the attraction of this elementary annulus in the direction pS

$$\begin{aligned} &= \frac{2\pi\rho r\delta r \cdot \phi u \cdot u\delta u}{b} \cdot \cos SpP, \\ &= \frac{2\pi\rho}{b} \cdot r\delta r \cdot u\phi u \cdot \delta u \left(\frac{b^2 - r^2 + u^2}{2bu} \right). \end{aligned}$$

Hence the attraction of the shell, which manifestly takes place in the direction pS ,

$$= \frac{\pi\rho}{b^2} \cdot \int_r \int_a r(b^2 - r^2 + u^2)\phi u.$$

The integral, with respect to u , is to be taken from $u = b - r$ to $b + r$, when the particle p is *without* the shell;

$$\begin{aligned}
 &= (\text{mass of the section through } N \text{ or } n) \cdot (pn \pm pN) \\
 &= (\text{the mass of a section}) \cdot 2pS \\
 &= (\text{the sum of the sections}) \cdot pS;
 \end{aligned}$$

(+ or - to be used according as the attracted particle is *without* or *within* the sphere.)

And the same being true of each pair of equidistant sections, it is clear the whole attraction of the shell

$$= (\text{mass of the shell}) \cdot pS, \text{ as before.}$$

13. Ex. 2. *When the attractive force of each particle varies inversely as the square of the distance.*

In this case $\phi u = \frac{1}{u^2}$, and

$$\begin{aligned}
 \therefore \int_r \int_u \frac{(b^2 - r^2 + u^2)}{u^2} &= \int_r \left\{ -\frac{b^2 r - r^3}{u} + ru + C \right\} \\
 &= 0 \text{ or } \int_r 4r^2 \\
 &= 0 \text{ or } \frac{4}{3}(r^3 + C') \\
 &= 0 \text{ or } \frac{4}{3}(a^3 - a'^3),
 \end{aligned}$$

according as the particle is *within* or *without* the shell.

Hence the attraction in the former case = 0, and in the latter

$$= \frac{4\pi\rho}{3b^2} \cdot (a^3 - a'^3) = \frac{\text{mass}}{b^2}.$$

14. The results of the last example, which is the case of Nature, may be arrived at in a different manner.

Using the notation of Art. 10, let $\zeta = \angle SpP$ (Fig. p. 329), B = the attraction of the shell in the direction pS ,

m = the mass of the annulus generated by PQ ;

$$\therefore m = 2\pi\rho r^2 \sin\theta \cdot \delta r \delta\theta;$$

$V = \Sigma \left(\frac{m}{u} \right)$ = the sum of the quotients of each particle divided by its distance from p .

Then

$$u^2 = b^2 - 2br \cos \theta + r^2,$$

$$\text{and } \therefore u d_\theta u = br \sin \theta,$$

$$\cos \zeta = \frac{b - r \cos \theta}{u} = d_b u;$$

$$\text{and } B = \Sigma \left(\frac{m}{u^2} \cdot \cos \zeta \right),$$

$$\begin{aligned} \text{also } -d_b V &= -\Sigma d_b \left(\frac{m}{u} \right) \\ &= \Sigma \left(\frac{m}{u^2} \cdot d_b u \right) \\ &= \Sigma \left(\frac{m}{u^2} \cdot \cos \zeta \right) \\ &= B. \end{aligned}$$

Now

$$\begin{aligned} V &= \Sigma \left(\frac{m}{u} \right) = 2\pi\rho \cdot \Sigma \left(\frac{r^2 \sin \theta \cdot \delta r \delta \theta}{u} \right) \\ &= 2\pi\rho \cdot \int_r \int_\theta \left(\frac{r^2 \sin \theta}{u} \right) \\ &= \frac{2\pi\rho}{b} \cdot \int_r \int_\theta (r d_\theta u), \text{ by eliminating } \sin \theta, \\ &= \frac{2\pi\rho}{b} \cdot \int_r (ru + C) \\ &= \frac{2\pi\rho}{b} \cdot \int_r 2r^2, \text{ when the particle is } \textit{without} \text{ the} \\ &\quad \text{shell,} \end{aligned}$$

$$\text{but } = \frac{2\pi\rho}{b} \cdot \int_r 2br, \text{ when it is } \textit{within} \text{ the shell}$$

$$= \frac{4\pi\rho}{3b} \cdot (a^3 - a'^3), \text{ and } 2\pi\rho \cdot (a^2 - a'^2) \text{ in the two cases.}$$

Hence

$$B = -d_b V = \frac{4\pi\rho}{3b^3} \cdot (a^3 - a'^3)$$

$$= \frac{\text{mass}}{b^3} \text{ in the former case;}$$

and = 0 in the latter case.

15. Since the attraction of a spherical shell on a particle is the same as if its mass were collected at its centre, both when the force of each particle varies as the distance, and when it varies inversely as the square of the distance, it follows that the attraction of a sphere composed of such particles will be the same as if it were collected at its centre, whether it be homogeneous or heterogeneous, providing the density be the same at all equal distances from the centre. For it may be considered as made up of an infinite number of concentric spherical shells, of indefinitely small thickness and uniform density, each of which produces the same effect as if it were collected at the centre.

16. When the force of each particle varies inversely as the square of the distance, the attractive force of a homogeneous sphere on a particle on its surface, found from Art. 13, by making $a = b$, and $a' = 0$, is $\frac{4}{3}\pi\rho b$, which therefore varies as the radius.

17. *To find the attractive force of a sphere on a particle situated within it.*

Conceive a concentric spherical surface described within the sphere, and passing through the attracted particle. The sphere is thus divided into a spherical shell, which, by Art. 13, produces no effect on the particle; and a smaller sphere on the surface of which the particle is situated, the attraction of which by the last Art.

$$= \frac{4}{3}\pi\rho \cdot (\text{radius})$$

$$= \frac{4}{3}\pi\rho \cdot (\text{distance of particle from the centre}).$$

This is, therefore, the attraction of the whole sphere; and it varies as the distance of the particle from the centre. See Art. 90, *Dynamics*.

18. *Two spheres attract each other in the same manner as if their masses were collected at their respective centres.*

For, by Art. 15, the sphere A will attract each particle of B in the same manner as if its whole mass were collected at its centre: and since, by Art. 29, *Dynamics*, the particle exerts an equal re-action on A , therefore each particle of B acts on A in the same manner as if A were collected at its centre; but B acts on a body, collected at a point, as if it were itself collected at its centre (Art. 15); and, therefore, A and B attract each other in the same manner as if they were respectively collected at their centres.

19. This is true when the force of each particle varies either as the distance or inversely as the square of the distance. Art. 15.

20. From Art. 13, the attraction of a shell of indefinitely small thickness δa

$$\begin{aligned} &= \frac{4\pi\rho}{3b^3} \{a^3 - (a - \delta a)^3\} \\ &= \frac{4\pi\rho}{b^3} \cdot a^2\delta a, \text{ ultimately.} \end{aligned}$$

21. *If a cone be described enveloping the shell just mentioned, and having its vertex in the attracted point, it will by the circle of contact divide the shell into two parts, (one convex, and the other concave, towards the attracted particle) whose attractions are equal.*

For, by Art. 10, the attraction of such a shell

$$\begin{aligned} &= \frac{\pi\rho}{b^3} \cdot \int_u \frac{a\delta a (b^2 - a^2 + u^2)}{u^2} \\ &= \frac{\pi\rho a\delta a}{b^3} \cdot \left(C - \frac{b^2 - a^2}{u} + u\right), \end{aligned}$$

and this taken from $u = b - a$ to $u = \sqrt{b^2 - a^2}$, gives

$$\frac{\pi\rho a\delta a}{b^3} \cdot 2a, \text{ or } \frac{2\pi\rho}{b^2} \cdot a^2\delta a$$

for the attraction of the convex part, which is half the attraction of the whole shell; and therefore the attractions of the convex and concave parts are equal.

22. Or we may demonstrate the same property thus;

$$\begin{aligned} V &= \Sigma \left(\frac{m}{u} \right) = 2\pi\rho \cdot \Sigma \left(\frac{a^2 \sin \theta \cdot \delta a \delta \theta}{u} \right) \\ &= \frac{2\pi\rho a \delta a}{b} \cdot \int_0^\theta d_\theta u, \text{ as in Art. 14,} \\ &= \frac{2\pi\rho a \delta a}{b} \cdot (u + C). \end{aligned}$$

This integral taken between $u = b - a$, and $u = b \sin a$, ($2a$ being the angle subtended at the centre of the shell by the convex part of the generating circle) will give

$$\begin{aligned} V &= \frac{2\pi\rho a \delta a}{b} \cdot (b \sin a - b + a) \\ &= 2\pi\rho a \delta a \cdot \left(\sin a - 1 + \frac{a}{b} \right) \end{aligned}$$

for the convex part; and, therefore, the attraction of this part

$$= -d_b V = \frac{2\pi\rho a^2 \delta a}{b^2}, \text{ as before.}$$

23. The following is *Laplace's method of finding the attraction of a spherical shell for any law of attraction of its component particles.*

Let ϕu be the attraction of a unit of matter at the distance u ; then, proceeding as in Art. 10, the attraction of the shell, whose internal radius is r and thickness δr , will be found to be

$$\begin{aligned} &= 2\pi\rho r^2 \delta r \cdot \int_0^\theta \left(\frac{b - r \cos \theta}{u} \cdot \phi u \cdot \sin \theta \right) \\ &= 2\pi\rho r^2 \delta r \cdot \int_0^\theta (\phi u \cdot d_\theta u \cdot \sin \theta), \text{ as in Art. 14.} \end{aligned}$$

Let now $\phi u \cdot d_b u = d_b \phi_1 u$, then the attraction

$$= 2\pi\rho r^2 \delta r \cdot \int_{\theta} (d_b \phi_1 u \cdot \sin \theta),$$

$$= 2\pi\rho r^2 \delta r \cdot d_b \int_{\theta} (\phi_1 u \cdot \sin \theta),$$

$$= 2\pi\rho r \delta r \cdot d_b \int_{\theta} \frac{u \phi_1 u \cdot d_{\theta} u}{b}, \text{ by eliminating } \sin \theta,$$

$$= 2\pi\rho r \delta r \cdot d_b \int_u \frac{u \phi_1 u}{b};$$

and if the integral of $u \phi_1 u$ be represented generally by $\psi u + C$, its value between the limits $u = b - r$ and $u = b + r$, will be

$$\int_u u \phi_1 u = \psi(b + r) - \psi(b - r);$$

and therefore the attraction of the shell on an external particle

$$= 2\pi\rho r \delta r \cdot d_b \left\{ \frac{\psi(b + r) - \psi(b - r)}{b} \right\}.$$

Similarly, its attraction on an internal particle

$$= 2\pi\rho r \delta r \cdot d_b \left\{ \frac{\psi(r + b) - \psi(r - b)}{b} \right\}.$$

24. *To find according to what law the particles of a spherical shell must attract, in order that the whole shell may attract an external particle in the same manner as if it were collected at its centre.*

By the last article, the attraction of the shell

$$= 2\pi\rho r \delta r \cdot d_b \left\{ \frac{\psi(b + r) - \psi(b - r)}{b} \right\}.$$

Now (ψ' , ψ'' , ... denoting $d_b \psi$, $d_b^2 \psi$ ) by ex-

panding these functions by Taylor's Theorem, the whole expression is reduced to (putting M for $4\pi\rho r^2\delta r$ the mass of the shell)

$$M \cdot \left\{ d_b \left(\frac{\psi' b}{b} \right) + d_b \left(\frac{\psi''' b}{b} \right) \cdot \frac{r^2}{2 \cdot 3} + \dots \right\},$$

which will be equal to the attraction of the whole collected at its centre, if

$$d_b \left(\frac{\psi''' b}{b} \right) = 0,$$

for then the terms containing r vanish. Hence, by integration of this equation, we obtain

$$\psi''' b = 3Ab,$$

and integrating again,

$$\psi'' b = \frac{3}{2} Ab^2 + B,$$

and integrating again,

$$\psi' b = \frac{1}{2} Ab^3 + Bb - C.$$

But $\psi' b = b\phi_1 b$;

$$\therefore b\phi_1 b = \frac{1}{2} Ab^3 + Bb - C;$$

$$\therefore \phi_1 b = \frac{1}{2} Ab^2 + B - \frac{C}{b},$$

$$\text{and } \therefore \phi b = d_b \phi_1 b = Ab + \frac{C}{b^2}.$$

that is, the force must vary either as the distance directly, or as the inverse square of the distance, or partly as one and partly as the other.

Hence the law of Nature is the only law of attraction of the particles of matter, that a spherical shell may attract with a force varying inversely as the square of the distance from its centre.

25. *To find the law of attraction of each particle, that a spherical shell may exert no attractive force on a particle within it.*

In this case, from Art. 23,

$$d_b \cdot \left\{ \frac{\psi(r+b) - \psi(r-b)}{b} \right\} = 0,$$

whatever be the value of b ; and therefore, by integrating, we must always have

$$\psi(r+b) - \psi(r-b) = -2Cb,$$

or expanding by Taylor's Theorem,

$$-Cb = \psi' r \cdot b + \psi''' r \cdot \frac{b^3}{1 \cdot 2 \cdot 3} + \dots;$$

$$\therefore \psi' r = -c;$$

$$\text{and } \therefore -C = \psi' b = b\phi_1 b;$$

$$\therefore \phi_1 b = -\frac{C}{b};$$

$$\therefore \phi b = d_b \phi_1 b = \frac{C}{b^2},$$

which is the law of Nature; and, therefore, by the last article, the force of each particle varies inversely as the square of the distance.

26. *To find the attraction of an oblate spheroid of uniform density and small ellipticity on a particle at its pole.*

Let a, b , be the semi major and minor axes of the spheroid, within which inscribe a sphere touching it at its poles. Then the equation of the generating ellipse is

$$y^2 = \frac{a^2}{b^2} (2bx - x^2),$$

the particle being the origin, and the axis of revolution the axis of x .

The mass of a circular annulus between the sphere and spheroid, which is contained between two planes parallel to the plane of the equator, at distances x and $x + \delta x$ from the particle

$$= \pi \left(1 - \frac{b^2}{a^2}\right) \rho \cdot y^2 \delta x, \text{ ultimately;}$$

or if $\epsilon = \frac{a-b}{a}$ (the ellipticity), then $\frac{b^2}{a^2} = 1 - 2\epsilon$ nearly, and the mass of the annulus

$$\begin{aligned} &= 2\pi\epsilon\rho \cdot \frac{a^2}{b^2} (2bx - x^2) \delta x, \\ &= 2\pi\epsilon\rho (2bx - x^2) \delta x, \text{ nearly.} \end{aligned}$$

Now every particle of this annulus may be considered at the same distance $\sqrt{2bx}$ from the attracted point; and, therefore, its attraction in the direction of the minor axis (in which direction the attraction of the whole spheroid takes place)

$$\begin{aligned} &= \frac{2\pi\epsilon\rho (2bx - x^2) \delta x}{2bx} \cdot \frac{x}{\sqrt{2bx}} \\ &= \frac{\pi\epsilon\rho}{b\sqrt{2b}} \cdot (2bx^{\frac{1}{2}} - x^{\frac{3}{2}}) \delta x; \end{aligned}$$

and by integrating this we have the attraction of the part which is exterior to the sphere

$$\begin{aligned} &= \frac{2\pi\epsilon\rho}{b\sqrt{2b}} \cdot \left(\frac{2}{3} bx^{\frac{3}{2}} - \frac{1}{5} x^{\frac{5}{2}}\right) + C, \\ &= \frac{16}{15} \pi\epsilon\rho b, \text{ between } x=0, \text{ and } x=2b. \end{aligned}$$

But the attraction of the sphere = $\frac{4}{3}\pi\rho b$ Art. 16, and therefore the attraction of the spheroid

$$= \frac{4}{3}\pi\rho b \left(1 + \frac{4}{5}\epsilon\right).$$

27. If b be greater than a , the spheroid will be prolate, and ϵ will be negative; and therefore the attraction of a prolate spheroid on a particle at its pole

$$= \frac{4}{3}\pi\rho b \left(1 - \frac{4}{5}\epsilon\right).$$

28. *To find the attraction of an oblate spheroid of uniform density and small ellipticity on a particle at its equator.*

Let the particle be taken for the origin, and the diameter of the equatorial circle passing through it for the axis of x ; and circumscribe a sphere about the spheroid.

Then the equation of the generating ellipse is

$$y^2 = \frac{b^2}{a^2} (2ax - x^2).$$

Now to find the attraction of that part of the sphere which is exterior to the spheroid, let us take a slice of it comprehended between two planes, at right angles to the axis of x , at the distances x and $x + \delta x$ from the attracted point. The mass of the sphere contained between these planes

$$= \pi \rho \frac{a^2}{b^2} y^2 \delta x,$$

and the mass of the spheroid between the same planes

$$= \pi \rho \frac{a}{b} y^2 \delta x,$$

and the difference of these

$$\begin{aligned} &= \pi \left(\frac{a^2}{b^2} - \frac{a}{b} \right) \rho y^2 \delta x \\ &= \pi \rho \epsilon (2ax - x^2) \delta x; \end{aligned}$$

and every particle of this mass may be considered at the same distance $\sqrt{2ax}$ from the attracted point; and therefore its attraction in the direction of the axis of x

$$\begin{aligned} &= \pi \rho \epsilon \cdot \frac{2ax - x^2}{2ax} \cdot \frac{x}{\sqrt{2ax}} \cdot \delta x \\ &= \frac{\pi \rho \epsilon}{2a\sqrt{2a}} \cdot (2ax^{\frac{1}{2}} - x^{\frac{3}{2}}) \cdot \delta x; \end{aligned}$$

and therefore the attraction of that part of the sphere which is exterior to the spheroid

$$= \frac{\pi \rho \epsilon}{a \sqrt{2a}} \cdot \left(\frac{2}{3} a x^{\frac{3}{2}} - \frac{1}{5} x^{\frac{5}{2}} \right) + C,$$

$$= \frac{8}{15} \pi \epsilon \rho a, \text{ between } x = 0, \text{ and } x = 2a;$$

and since the attraction of the sphere $= \frac{4}{3} \pi \rho a$;

$\therefore$ the attraction of the spheroid

$$= \frac{4}{3} \pi \rho a \left(1 - \frac{2}{5} \epsilon \right).$$

29. If b be greater than a , the spheroid will be prolate, and ϵ will be negative, and the attraction of a prolate spheroid on a particle at its equator

$$= \frac{4}{3} \pi \rho a \left(1 + \frac{2}{5} \epsilon \right).$$

30. *To find the attraction of an oblate spheroid of uniform density on a particle at its pole.*

Taking the attracted point as origin, and putting r for the distance of a small element from that point, and θ the angle which r makes with the axis of revolution, a small element of the generating ellipse $= r \delta r \delta \theta$, and the mass generated by the revolution of this element

$$= 2\pi \rho r^2 \delta r \cdot \sin \theta \cdot \delta \theta;$$

and the attraction of this annulus in the direction of the axis of x

$$= 2\pi \rho \delta r \cdot \sin \theta \cos \theta \cdot \delta \theta;$$

and therefore the attraction of the spheroid

$$= 2\pi \rho \cdot \int_r \int_\theta \sin \theta \cos \theta$$

$$= 2\pi \rho \cdot \int_\theta (r + C) \sin \theta \cos \theta;$$

the integral is to be taken from $r = 0$, to $r = \frac{2ba^2 \cos \theta}{b^2 + a^2 \epsilon^2 \cos^2 \theta}$;

this value of r being determined from the equation of the generating ellipse $y^2 = \frac{a^2}{b^2} (2bx - x^2)$, by writing $r \sin \theta$ and $r \cos \theta$ respectively for y and x , and e for the eccentricity.

Hence the attraction

$$\begin{aligned}
 &= \frac{2\pi\rho}{e^2} \cdot \int_0^{2b} \frac{a^2 e^2 \cos^2 \theta \sin \theta}{b^2 + a^2 e^2 \cos^2 \theta} \\
 &= \frac{4b\pi\rho}{e^2} \cdot \int_0^1 \left\{ \sin \theta - \frac{b^2 \sin \theta}{b^2 + a^2 e^2 \cos^2 \theta} \right\} \dots\dots (1) \\
 &= \frac{4b\pi\rho}{e^2} \cdot \left\{ C - \cos \theta + \frac{b}{ae} \tan^{-1} \left(\frac{ae \cos \theta}{b} \right) \right\} \\
 &= \frac{4b\pi\rho}{e^2} \cdot \left(1 - \frac{b}{ae} \tan^{-1} \frac{ae}{b} \right) \\
 &= 4b\pi\rho \left(\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^2} \sin^{-1} e \right);
 \end{aligned}$$

the integral being taken from $\theta = 0$, to $\theta = \frac{\pi}{2}$, for the whole spheroid.

31. If the spheroid should be prolate we must write a for b and b for a in equation (1) of last article. And for $\frac{b}{e^2}$ which is equal to $\frac{a^2 b}{a^2 - b^2}$ we must write $\frac{b^2 a}{b^2 - a^2}$ or $-\frac{b^2}{ae^2}$; and for $a^2 e^2$ or $a^2 - b^2$, we must write $b^2 - a^2$, or $-a^2 e^2$, consequently equation (1) becomes

the attraction of a prolate spheroid

$$\begin{aligned}
 &= -\frac{4b^3\pi\rho}{ae^2} \int_0^1 \left\{ \sin \theta - \frac{a^2 \sin \theta}{a^2 - a^2 e^2 \cos^2 \theta} \right\} \\
 &= \frac{4b^3\pi\rho}{ae^2} \int_0^1 \left\{ \frac{\sin \theta}{1 - e^2 \cos^2 \theta} - \sin \theta \right\} \\
 &= \frac{4b^3\pi\rho}{ae^2} \left\{ -\frac{1}{2e} \log_e \frac{1 + e \cos \theta}{1 - e \cos \theta} + \cos \theta + C \right\} \\
 &= \frac{4b^3\pi\rho}{ae^2} \left\{ \frac{1}{2e} \log_e \frac{1 + e}{1 - e} - 1 \right\}, \text{ between limits.}
 \end{aligned}$$

32. *To find the attraction of an oblate spheroid of uniform density on a particle at its equator.*

Let the particle be taken for the origin; the equator for the plane xy ; and that diameter of it which passes through the particle for the axis of x ; r the distance of an element of the spheroid from the origin; θ the inclination of r to the plane xy ; ϕ the inclination of its projection on that plane to the axis of x .

Then an element of the mass $= \rho r^2 \delta r \cdot \cos \theta \cdot \delta \theta \cdot \delta \phi$; and the attraction of this element in the direction of the axis of x , (which is evidently the direction in which the whole attraction of the spheroid takes place)

$$= \rho \delta r \cdot \cos \theta \cdot \delta \theta \delta \phi \cdot \cos \theta \cos \phi;$$

and, therefore, the attraction of the spheroid

$$\begin{aligned} &= \rho \cdot \int_r \int_\theta \int_\phi \cos^2 \theta \cos \phi; \\ &= \rho \cdot \int_\theta \int_\phi (r + C) \cos^2 \theta \cos \phi \\ &= \frac{2\rho b^3}{ae^2} \cdot \int_\theta \int_\phi \frac{e^2 \cos^2 \theta \cos^2 \phi}{1 - e^2 \cos^2 \theta}, \end{aligned}$$

(the integral being taken from $r = 0$, to $r = \frac{2b^2 \cos \theta \cos \phi}{a(1 - e^2 \cos^2 \theta)}$; this value of r being deduced from the equation of the spheroidal surface $x^2 + y^2 + \frac{a^2 z^2}{b^2} = 2ax$, by writing $r \cos \theta \cos \phi$, $r \cos \theta \sin \phi$, and $r \sin \theta$ respectively for x , y and z)

$$\begin{aligned} &= \frac{2\rho b^3}{ae^2} \int_\theta \int_\phi \cdot \cos^2 \phi \left\{ \frac{\cos \theta}{1 - e^2 + e^2 \sin^2 \theta} - \cos \theta \right\} \\ &= \frac{2\rho b^3}{ae^2} \int_\theta \int_\phi \cdot \cos^2 \phi \left\{ \frac{a^2 \cos \theta}{b^2 + a^2 e^2 \sin^2 \theta} - \cos \theta \right\} \dots\dots (1) \\ &= \frac{2\rho b^3}{ae^2} \cdot \int_\phi \cdot \cos^2 \phi \left\{ \frac{a}{be} \cdot \tan^{-1} \frac{ae \sin \theta}{b} - \sin \theta + C \right\} \\ &= \frac{2\rho b^3}{ae^2} \cdot \int_\phi (1 + \cos 2\phi) \left(\frac{a}{be} \cdot \sin^{-1} e - 1 \right), \end{aligned}$$

(the integral being taken between $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$)

$$\begin{aligned} &= \frac{2\rho b^2}{ae^2} \left(\phi + \frac{1}{2} \sin 2\phi \right) \left(\frac{a}{be} \cdot \sin^{-1} e - 1 \right) + C \\ &= 2\pi\rho b \cdot \left(\frac{\sin^{-1} e}{e^2} - \frac{\sqrt{1-e^2}}{e^2} \right), \end{aligned}$$

the integral being taken between $\phi = -\frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$.

33. If the spheroid be prolate we must proceed with equation (I) as in Art. 31; writing $-\frac{b}{e^2}$ for $\frac{b^2}{ae^2}$ and $-a^2e^2$ for a^2e^2 . By these substitutions, we obtain the attraction of a prolate spheroid

$$\begin{aligned} &= -\frac{2\rho b}{e^2} \int_0^\phi \cos^2 \phi \left\{ \frac{b^2 \cos \theta}{a^2 - a^2 e^2 \sin^2 \theta} - \cos \theta \right\} \\ &= -\frac{2\rho b}{e^2} \int_0^\phi \cos^2 \phi \left\{ \frac{b^2}{2a^2 e} \log_e \frac{1+e \sin \theta}{1-e \sin \theta} - \sin \theta + C \right\} \\ &= -\frac{2\rho b}{e^2} \int_0^\phi \cos^2 \phi \left\{ \frac{b^2}{a^2 e} \log_e \frac{1+e}{1-e} - 2 \right\}, \text{ between } \theta = \mp \frac{\pi}{2} \\ &= -\frac{\rho b}{e^2} \int_0^\phi (1 + \cos 2\phi) \left\{ \frac{b^2}{a^2 e} \log_e \frac{1+e}{1-e} - 2 \right\} \\ &= \frac{\pi\rho b}{e^2} \left\{ 2 - \frac{b^2}{a^2 e} \log_e \frac{1+e}{1-e} \right\} \\ &= \pi\rho b \left(\frac{2}{e^2} - \frac{1-e^2}{e^3} \log_e \frac{1+e}{1-e} \right). \end{aligned}$$

34. If V be the sum of the quotients of the particles of a body divided by their respective distances from a given point, whose co-ordinates are f, g, h ; then the attractions F, G, H of the body on a particle situated at that point in the directions of the co-ordinates, are respectively equal to

$$-d_f V, -d_g V, -d_h V;$$

and V may be determined from the equation

$$d_f^2 V + d_g^2 V + d_h^2 V = 0;$$

$$\text{or, } d_f^2 V + d_g^2 V + d_h^2 V = -4\pi\rho,$$

the latter only being true when the particle attracted is a part of the body.

For, let x, y, z be the co-ordinates of an element m of the body, and u = its distance from the attracted point; then

$$u^2 = (f - x)^2 + (g - y)^2 + (h - z)^2,$$

$$\text{and } V = \Sigma \left(\frac{m}{u} \right);$$

$$\text{and } \therefore d_f V = \Sigma \left(d_f \frac{m}{u} \right) = - \Sigma \left(\frac{m}{u^2} \cdot d_f u \right)$$

$$= - \Sigma \left(\frac{m}{u^2} \cdot \frac{f - x}{u} \right) = -F.$$

Similarly, $d_g V = -G$, and $d_h V = -H$.

Again, $u d_f u = f - x$;

and, differentiating again,

$$u d_f^2 u + (d_f u)^2 = 1;$$

$$\therefore d_f^2 u = \frac{1}{u} - \frac{(d_f u)^2}{u}.$$

$$\text{Now } d_f^2 V = -d_f \Sigma \left(\frac{m}{u^2} \cdot d_f u \right)$$

$$= \Sigma \left(\frac{2m}{u^3} \cdot (d_f u)^2 - \frac{m}{u^2} \cdot d_f^2 u \right)$$

$$= \Sigma \left(\frac{3m}{u^3} \cdot d_f u \cdot d_f u - \frac{m}{u^3} \right).$$

$$\text{Similarly, } d_g^2 V = \Sigma \left(\frac{3m}{u^3} \cdot (d_g u)^2 - \frac{m}{u^3} \right),$$

$$\text{and } d_h^2 V = \Sigma \left(\frac{3m}{u^3} \cdot (d_h u)^2 - \frac{m}{u^3} \right);$$

$$\therefore d_f^2 V + d_g^2 V + d_h^2 V = 0 \dots \dots (1),$$

because

$$(d_f u)^2 + (d_g u)^2 + (d_h u)^2 = \left(\frac{f-x}{u} \right)^2 + \left(\frac{g-y}{u} \right)^2 + \left(\frac{h-z}{u} \right)^2 = 1.$$

35. Equation (1) is universally true when the particle attracted is not a part of the attracting body; it fails, however, in the contrary case, for then u becomes evanescent between the limits of the sum of $\left(\frac{m}{u} \right)$ for the whole body, and consequently there is a term in the expression for $d_f^2 V + d_g^2 V + d_h^2 V$ of the form $\frac{0}{0}$. To determine the proper form of equation

(1), in this case, let a spherical surface having its centre in the origin of co-ordinates be inscribed within the body so as to include the particle attracted within the mass of the sphere thus formed; the function V will thus be separated into two parts U , U' ; of which U belongs to the sphere, and U' to the part of the body exterior to the sphere; since, therefore, the attracted particle is not a portion of this latter body, we have, by the former part of the investigation

$$d_f^2 U' + d_g^2 U' + d_h^2 U' = 0;$$

and, therefore,

$$d_f^2 V + d_g^2 V + d_h^2 V = d_f^2 U + d_g^2 U + d_h^2 U.$$

But the three quantities $-d_f U$, $-d_g U$, $-d_h U$, being the attractions which the sphere exercises on the particle in the directions of the co-ordinates, will be respectively equal to $\frac{4}{3}\pi\rho f$, $\frac{4}{3}\pi\rho g$, $\frac{4}{3}\pi\rho h$, by Art. 17; that is,

$$d_f U = -\frac{4}{3}\pi\rho f,$$

$$d_g U = -\frac{4}{3}\pi\rho g,$$

$$d_h U = -\frac{4}{3}\pi\rho h.$$

Differentiating each of these equations, we have

$$d_f^2 U = -\frac{4}{3}\pi\rho,$$

$$d_g^2 U = -\frac{4}{3}\pi\rho,$$

$$d_h^2 U = -\frac{4}{3}\pi\rho;$$

and, therefore,

$$d_f^2 V + d_g^2 V + d_h^2 V = -4\pi\rho\dots\dots(2).$$

36. As an example of the application of equation (1), suppose it were required to find the attraction of a sphere on an external particle.

Let the centre of the sphere be the origin of co-ordinates, and $b^2 = f^2 + g^2 + h^2$. Then, from the symmetrical form of a sphere, so long as b remains the same, the attraction is evidently the same; and therefore V is a function of b only; and fgh enter V , only because they enter b ; consequently

$$d_f V = d_b V \cdot d_f b = d_b V \cdot \frac{f}{b};$$

$$\begin{aligned} \therefore d_f^2 V &= \frac{1}{b} d_b V + f d_f \left(\frac{d_b V}{b} \right), \\ &= d_b^2 V \cdot \frac{f^2}{b^3} + \frac{1}{b} \cdot d_b V - d_b V \cdot \frac{f^2}{b^3}. \end{aligned}$$

$$\text{Similarly, } d_g^2 V = d_b^2 V \cdot \frac{g^2}{b^3} + \frac{1}{b} \cdot d_b V - d_b V \cdot \frac{g^2}{b^3},$$

$$\text{and } d_h^2 V = d_b^2 V \cdot \frac{h^2}{b^3} + \frac{1}{b} \cdot d_b V - d_b V \cdot \frac{h^2}{b^3};$$

$$\begin{aligned} \therefore 0 &= d_f^2 V + d_g^2 V + d_h^2 V \\ &= d_b^2 V + \frac{2}{b} \cdot d_b V. \end{aligned}$$

Multiplying by b^2 , integrating once, and then dividing by b^2 , we find the attraction of the sphere

$$= -d_b V = \frac{A}{b^3}.$$

37. As an application of equation (2), suppose the particle to be within the sphere.

$$\text{Then } -4\pi\rho = d_b^2 V + \frac{2}{b} d_b V.$$

Multiplying by b^2 , and integrating, we have

$$A - \frac{4}{3}\pi\rho b^3 = b^2 d_b V.$$

But when $b = 0$, $d_b V = 0$, and $\therefore A = 0$; and, consequently, the attraction of a sphere on a point within it

$$= -d_b V = \frac{4}{3}\pi\rho b.$$

38. Again, suppose it were required to find the attraction of a cylinder of indefinite length on an external particle.

It is evident, from the symmetrical form of the cylinder, and from its being of indefinite length, that the attraction will remain the same so long as b , the distance from its axis, is the same. Now, supposing the axis of the cylinder to be the axis of z , $b^2 = f^2 + g^2$; and V is a function of b only; and f, g enter V , only because they exist in b ; and h does not enter V at all,

$$\therefore d_f^2 V + d_g^2 V = 0.$$

$$\text{But } d_f^2 V = d_b^2 V \cdot \frac{f^2}{b^2} + \frac{1}{b} \cdot d_b V - d_b V \cdot \frac{f^2}{b^3},$$

$$\text{and } d_g^2 V = d_b^2 V \cdot \frac{g^2}{b^2} + \frac{1}{b} \cdot d_b V - d_b V \cdot \frac{g^2}{b^3};$$

$$\therefore 0 = d_f^2 V + d_g^2 V = d_b^2 V + \frac{1}{b} \cdot d_b V.$$

Multiplying by b , and integrating, we have the attraction

$$= -d_b V = \frac{A}{b}.$$

Consequently the attraction of a cylinder of indefinite length varies inversely as the distance from its axis.

39. If the particle be within the cylinder,

$$\text{then } -4\pi\rho = d_b^2 V + \frac{1}{b} \cdot d_b V;$$

$$\therefore A - 2\pi\rho b^2 = b \cdot d_b V.$$

But when $b = 0$, $d_b V = 0$; and $\therefore A = 0$.

Hence the attraction of an indefinite cylinder on one of its own particles

$$= -d_b V = 2\pi\rho b.$$

40. Lastly, suppose it were *required to find the attraction of a solid, of indefinite extent, on an external particle; the solid being bounded on the side next to the particle by a plane.*

Here V is a function of b (the distance from the surface) only, and $b = f$; g and h do not enter b ;

$$\therefore d_b^2 V = 0.$$

Consequently, the attraction ($-d_b V$) is constant.

ON THE ATTRACTION OF AN ELLSIPSOID.

41. Let the co-ordinate axes of x, y, z , coincide with the semi-axes a, b, c of the ellipsoid; let f, g, h be the co-ordinates of the attracted particle, which take for the origin of polar co-ordinates; u the distance of an element m of the ellipsoid from this origin; θ the angle which u makes with the axis of x ; and ϕ the angle which its projection on the plane yx makes with the axis of y . Then m may be considered as a small rectangular parallelopiped whose dimensions are $\delta u, u\delta\theta, u\sin\theta\delta\phi$;

$$\text{and } \therefore m = u^2 \sin \theta \cdot \delta u \delta \theta \delta \phi,$$

considering the density uniform, and equal to 1.

Consequently,

$$F = \Sigma \left(\frac{m}{u^2} \cdot \cos \theta \right) = \int_u \int_\theta \int_\phi \sin \theta \cos \theta.$$

Similarly,

$$G = \int_u \int_\theta \int_\phi \sin^2 \theta \cos \phi, \text{ and } H = \int_u \int_\theta \int_\phi \sin^2 \theta \sin \phi.$$

In integrating with respect to u , we observe that two cases occur, according as the particle is *within* or *without* the ellipsoid.

42. In the former case, the line which passes through the particle attracted, is divided into two parts u, u' , which are terminated by the surface of the ellipsoid; the integrals are therefore to be taken between these limits, and consequently

$$F = \int_\theta \int_\phi (u' + u) \sin \theta \cos \theta,$$

$$G = \int_\theta \int_\phi (u' + u) \sin^2 \theta \cos \phi,$$

$$H = \int_\theta \int_\phi (u' + u) \sin^2 \theta \sin \phi.$$

These expressions are now to be successively integrated, with regard to θ and ϕ , from θ and ϕ each = 0, to each = π .

43. In the latter case, u cuts the ellipsoid in two points, the distances of which from the particle attracted are u and u' ; the integrals are therefore to be taken between these limits; consequently

$$F = \int_\theta \int_\phi (u' - u) \sin \theta \cos \theta,$$

$$G = \int_\theta \int_\phi (u' - u) \sin^2 \theta \cos \phi,$$

$$H = \int_\theta \int_\phi (u' - u) \sin^2 \theta \sin \phi.$$

These expressions are now to be integrated between the values of θ and ϕ , corresponding to the points where $u' - u = 0$, that is, where u is a tangent to the ellipsoid.

44. Now $x = f - u \cos \theta$, $y = g - u \sin \theta \cos \phi$, $z = h - u \sin \theta \sin \phi$; and substituting these in the equation of the

surface of the ellipsoid, in order to determine the values of u and u' , we have for that purpose the equation

$$\begin{aligned} & \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta \cos^2 \phi}{b^2} + \frac{\sin^2 \theta \sin^2 \phi}{c^2} \right) u^2 - \\ & \left(\frac{f \cos \theta}{a^2} + \frac{g \sin \theta \cos \phi}{b^2} + \frac{h \sin \theta \sin \phi}{c^2} \right) 2u \\ & = 1 - \frac{f^2}{a^2} - \frac{g^2}{b^2} - \frac{h^2}{c^2}; \end{aligned}$$

from which, by writing L and M for the coefficients of u^2 and $-2u$, and N for $M^2 + L \left(1 - \frac{f^2}{a^2} - \frac{g^2}{b^2} - \frac{h^2}{c^2} \right)$, we obtain

$$u' = \frac{M + \sqrt{N}}{L}, \text{ and } u = \frac{M - \sqrt{N}}{L}.$$

Consequently, when the particle is *within* the ellipsoid,

$$\left. \begin{aligned} F &= 2 \int_{\theta} \int_{\phi} \left(\frac{M}{L} \cdot \sin \theta \cos \theta \right), \\ G &= 2 \int_{\theta} \int_{\phi} \left(\frac{M}{L} \cdot \sin^2 \theta \cos \phi \right), \\ H &= 2 \int_{\theta} \int_{\phi} \left(\frac{M}{L} \cdot \sin^2 \theta \sin \phi \right), \end{aligned} \right\} \dots\dots (A),$$

and when the particle is *without* the ellipsoid,

$$\begin{aligned} F &= 2 \int_{\theta} \int_{\phi} \left(\frac{\sqrt{N}}{L} \cdot \sin \theta \cos \theta \right), \\ G &= 2 \int_{\theta} \int_{\phi} \left(\frac{\sqrt{N}}{L} \cdot \sin^2 \theta \cos \phi \right), \\ H &= 2 \int_{\theta} \int_{\phi} \left(\frac{\sqrt{N}}{L} \cdot \sin^2 \theta \sin \phi \right). \end{aligned}$$

45. On account of the irrational quantity $\sqrt{N}$, it is quite impossible to integrate these latter expressions by any

known method. We are able, however, to elude the difficulty, by making the attraction of the ellipsoid on an *external* particle depend upon the attraction of another ellipsoid on an *internal* particle, by a process, first discovered by Mr. Ivory, which the reader will find detailed in Airy's *Heterogeneous Figure of the Earth*. For this reason, we shall confine ourselves to the determination of the attraction of an ellipsoid on an *internal* particle.

46. For this purpose, substituting for M its value in the first of equations (A), we find,

$$F = \frac{2f}{a^2} \int_{\theta} \int_{\phi} \frac{\sin \theta \cos^2 \theta}{L} + \frac{2g}{b^2} \int_{\theta} \int_{\phi} \frac{\sin^2 \theta \cos \theta \cos \phi}{L} + \frac{2h}{c^2} \int_{\theta} \int_{\phi} \frac{\sin^2 \theta \cos \theta \sin \phi}{L}.$$

Now, it is well known, if P be any rational function whatever of $\sin \theta$ and $\cos^2 \theta$, that $\int_{\theta} P \cos \theta = 0$ between the limits $\theta = 0$ and $\theta = \pi$; hence the last two terms in the above expression disappear by integration; and, consequently, writing its value for L , we have

$$F = 2f \cdot \int_{\theta} \int_{\phi} \frac{\sin \theta \cos^2 \theta}{\cos^2 \theta + \frac{a^2}{b^2} \sin^2 \theta \cos^2 \phi + \frac{a^2}{c^2} \sin^2 \theta \sin^2 \phi}.$$

Similarly,

$$G = 2g \cdot \int_{\theta} \int_{\phi} \frac{\sin^3 \theta \cos^2 \phi}{\sin^2 \theta \cos^2 \phi + \frac{b^2}{a^2} \cos^2 \theta + \frac{b^2}{c^2} \sin^2 \theta \sin^2 \phi},$$

and

$$H = 2h \cdot \int_{\theta} \int_{\phi} \frac{\sin^3 \theta \sin^2 \phi}{\sin^2 \theta \sin^2 \phi + \frac{c^2}{a^2} \cos^2 \theta + \frac{c^2}{b^2} \sin^2 \theta \cos^2 \phi}.$$

47. Dividing these three equations respectively by f , g and h , and adding the results, we obtain

$$\frac{F}{f} + \frac{G}{g} + \frac{H}{h} = 2 \int_0^\pi \int_0^\pi \sin \theta = 4\pi,$$

a very remarkable equation.

48. The values of F , G , and H , contain only the ratios of the quantities a, b, c ; so long, therefore, as these ratios remain the same, F, G, H , will remain unaltered; whence we conclude that all similar ellipsoids exert the same attraction on an internal particle whose co-ordinates are given.

49. And from this it follows, that an ellipsoidal shell, contained between similar concentric ellipsoidal surfaces, exerts no attraction on a particle within it; because it is the difference between two similar and concentric ellipsoids, which, by last Art., exert equal attractions on the particle.

50. The integrals in the last expressions for F, G and H , are to be taken between limits (see Art. 42.) which are independent of f, g and h , and therefore the integrals themselves are independent of these quantities; consequently

$$F \propto f, \quad G \propto g, \quad H \propto h.$$

51. Hence F remains the same so long as f is unaltered; let, therefore, another particle be placed in such a situation that g and h may vanish, but f remain the same as before; then, since the attraction of the ellipsoidal shell, which is contained between the external surface, and a similar concentric surface passing through the second attracted particle, is evanescent (Art. 49), we infer that the attraction of the whole ellipsoid on the first particle is equal to the attraction of the small ellipsoid, contained by the second surface, on the second particle, in the same direction. (See Airy's Tracts, *Homogeneous Figure of the Earth*, Prop. 8).

52. Integrating now the expression for F in Art. 46, with regard to ϕ , between the limits $\phi = 0$, and $\phi = \pi$, we find

$$F = 2f\pi \cdot \int_0^\pi \frac{\sin \theta \cos^2 \theta}{\left(\cos^2 \theta + \frac{a^2}{b^2} \sin^2 \theta\right)^{\frac{3}{2}} \cdot \left(\cos^2 \theta + \frac{a^2}{b^2} \sin^2 \theta\right)^{\frac{3}{2}}} d\theta.$$

The integral of this is to be taken from $\theta = 0$, to $\theta = \pi$, which comes to the same as taking it from $\theta = 0$, to $\theta = \frac{\pi}{2}$, and doubling the result. Hence, by writing v for $\cos \theta$, and μ for the mass of the ellipsoid, we obtain

$$F = \frac{3\mu f}{a^3} \cdot \int_0^1 \frac{v^3}{\left\{1 + \left(\frac{b^2}{a^2} - 1\right)v^2\right\}^{\frac{1}{2}} \cdot \left\{1 + \left(\frac{c^2}{a^2} - 1\right)v^2\right\}^{\frac{1}{2}}},$$

the integral of which is to be taken from $v = 0$, to $v = 1$.

53. If this integration could be effected, it is evident that v would disappear in the result; and therefore F is a function of f , a , b , c only; consequently G and H are *similar* functions of g , b , a , c ; and h , c , a , b ; respectively.

$$\therefore G = \frac{3\mu g}{b^3} \cdot \int_0^1 \frac{v^3}{\left\{1 + \left(\frac{a^2}{b^2} - 1\right)v^2\right\}^{\frac{1}{2}} \cdot \left\{1 + \left(\frac{c^2}{b^2} - 1\right)v^2\right\}^{\frac{1}{2}}},$$

$$\text{and } H = \frac{3\mu h}{c^3} \cdot \int_0^1 \frac{v^3}{\left\{1 + \left(\frac{b^2}{c^2} - 1\right)v^2\right\}^{\frac{1}{2}} \cdot \left\{1 + \left(\frac{a^2}{c^2} - 1\right)v^2\right\}^{\frac{1}{2}}},$$

between the same limits of v as before.

54. It is impossible to exhibit the integral, with regard to v , in finite terms; we may, however, put the values of F , G and H under a somewhat more convenient form, as follows.

For $\frac{b^2}{a^2} - 1$, and $\frac{c^2}{a^2} - 1$, substitute e^2 and e'^2 respectively;

and in the value of G write $\frac{bv'}{a\sqrt{1 + e'^2v'^2}}$ for v ; and in that of

H write $\frac{cv''}{a\sqrt{1 + e^2v''^2}}$ for v ; then

$$F = \frac{3\mu f}{a^3} \cdot \int_0^1 \frac{v^3}{\sqrt{1 + e^2v^2} \cdot \sqrt{1 + e'^2v^2}},$$

$$G = \frac{3\mu g}{a^3} \int_v \frac{v'^2}{(1 + e^2 v'^2)^{\frac{3}{2}} \cdot (1 + e^2 v''^2)^{\frac{3}{2}}},$$

$$H = \frac{3\mu h}{a^3} \cdot \int_{v''} \frac{v''^2}{\sqrt{1 + e^2 v''^2} \cdot (1 + e^2 v''^2)^{\frac{3}{2}}}.$$

Since, however, these integrals are all to be taken between $v = 0$ and $v = 1$, and at these limits v' and v'' have the same values as v , we may in the expressions for G and H write v for v' and v'' without altering the final results. This being done, and substituting

$$F' \text{ for } \int_v \frac{v^2}{\sqrt{1 + e^2 v^2} \cdot \sqrt{1 + e^2 v^2}} \left\{ \begin{array}{l} \text{from } v = 0 \\ \text{to } v = 1 \end{array} \right\},$$

we have

$$F = \frac{3\mu f}{a^3} \cdot F',$$

$$G = \frac{3\mu g}{a^3} \cdot d_e(eF'),$$

$$H = \frac{3\mu h}{a^3} \cdot d_e(eF').$$

55. We can find F' in finite terms only in two cases, when $e = e$, and when one of them $= 0$; on both which suppositions two of the three semi-axes a , b , c are equal; and, consequently, the ellipsoid becomes a spheroid of revolution, the attraction of which may be found independently of F' , by Arts 30—33, 51.

56. For a more extended view of the subject of the Attraction of Spheroids, and for the application of the results above deduced to the determination of the figures of the Earth, and the heavenly bodies, the reader is referred to the writings of Newton and Laplace; or to *Airy's Mathematical Tracts*, and the *Théorie Analytique du Système du Monde*, tome ii. of Pontécoulant.

MISCELLANEOUS PROBLEMS.

1. A row of four elastic balls A, B, C, D is placed in a right line. Required the ratios of their masses, that the momentum of A may after impact be equally divided among the four; B, C, D being originally at rest.

Find the result also when there are n balls.

2. Two billiard balls are lying in contact on the table; in what direction must one of them be struck by a third so as to go off in a given direction? The balls are supposed equal in all respects, smooth, and of given elasticity.

3. AB is the vertical diameter of a circle. A perfectly elastic ball descends down the chord AC , and being reflected by the plane BC describes its path as a projectile; shew that the body will strike the circle at the opposite extremity of the diameter CD .

4. If two bodies be projected from the same point, at the same instant, with velocities v and v_1 , and in directions α and α_1 , shew that the time which elapses between their transits through the other point which is common to both their paths is equal

$$\text{to } \frac{2vv_1 \sin(\alpha - \alpha_1)}{g(v \cos \alpha + v_1 \cos \alpha_1)}.$$

5. An imperfectly elastic body, acted on by gravity, is projected in any direction with a velocity V against a vertical plane, making an angle θ with the vertical plane in which the body moves; find the velocity V' after impact; and shew that if the ball were perfectly elastic, α the elevation which would make V' a minimum, and a the distance of the point

of projection from the given plane $\tan \alpha = \frac{V^2 \sin \theta}{ga}$, and $V' = V \cot \alpha$.

6. A perfectly elastic ball falls from a height h on a plane inclined 30° to the horizon; shew that it will strike the plane again after an interval equal to twice the time of its fall, and that its range on the plane will be $4h$.

7. Find the locus of all the points to which a body will descend in the same time on straight lines drawn from two given points.

8. Find the line of quickest descent from the focus to the curve of a parabola, whose axis is vertical, and vertex upwards.

9. If a ball, whose elasticity is λ , be projected in vacuo at an elevation α , with the velocity u . Shew that it will go to the distance $\frac{u^2}{g} \cdot \frac{\sin 2\alpha}{1-\lambda}$ on a smooth horizontal plane before it ceases to rebound. Does the ball stop there?

10. A body, of the elasticity λ , is projected up an inclined plane with a given velocity, and at the top impinges perpendicularly upon another plane, and returns to the point from which it set out with two thirds of the velocity of projection. Find the length of the plane, and shew that λ is less than $\frac{2}{3}$.

11. A ball, of the elasticity λ , is projected with a velocity u at an angle of elevation α ; and at the instant of attaining its greatest altitude, it strikes a similar and equal ball falling vertically with a velocity $\frac{u}{2}$ at that point. Shew that the distance of the balls from each other at the end of the time t after the impact is $\frac{1}{2}tu\sqrt{1+4\lambda^2\cos^2\alpha}$.

12. A body falls from rest acted on by a central force varying as $\frac{1}{\sqrt{\text{Dist.}}}$; it is required to find the time of describing a given space.

13. If a particle in space be acted on only by a force which is parallel to the axis of x , shew that its intensity is equal to

$$a^2 d_x^2 x + 2ab d_x d_y x + b^2 d_y^2 x;$$

where a and b are the constant velocities parallel to the axes of x and y respectively.

14. Find the law of force by which a body may describe a lemniscate; the centre of force being the pole of the lemniscate.

15. A body acted upon by two central forces, each varying inversely as the square of the distance, is projected from a point between them towards one of the centres: required the velocity of projection that the body may just arrive at the neutral point of attraction and remain at rest there.

16. Two imperfectly elastic spheres attract one another with forces which $\propto D^{-2}$; find the greatest separation of their centres after n impacts, their original distance being given.

17. Three equal particles repelling each other with forces $\propto D^{-2}$ are placed in the same straight line, the two extremes being immovable. Find the motion of the one which is interposed.

18. Apply the equations of motions to shew that if a body describe a circle, of radius R , by the action of a force F tending to its centre; the $(\text{vel.})^2 = FR$.

19. A particle attached to one end of an elastic string moves on a smooth horizontal plane, the other end of the string being fixed to a point in the plane. If the path of the particle be a circle, shew that periodic time $\propto \left(\frac{ra}{r-a}\right)^{\frac{1}{2}}$, a and r being the natural and stretched lengths of the string. If the orbit be nearly circular find the angle between the apsides.

20. Find the law of force by which a body may describe a circle: the centre of force being in the circumference of the

circle. Find also the periodic time, and compare it with the period in the same circle when the same force is in its centre.

21. A body, acted on by a force varying inversely as the fifth power of the distance, is projected in any direction with a velocity equal to that which would be acquired in falling from an infinite distance: find the orbit.

22. A particle acted on by a central force varying as $\frac{1}{D^3}$, is projected from a given point with a velocity which is to the velocity from infinity as 5 : 3; in a direction making an angle $\sin^{-1} \frac{2\sqrt{6}}{5}$ with the radius vector. Find the position of the apses, and the time of describing a given angle.

23. Find the time in which a body will move from the vertex to the end of the latus rectum of a parabola; and shew that if the velocity be there suddenly altered in the ratio $m : 1$ (m being < 1) the body will proceed to describe an ellipse the eccentricity of which $= (1 - 2m^2 + 2m^4)^{\frac{1}{2}}$.

24. If the earth's orbit be taken an exact circle, and a comet be supposed to describe round the sun a parabolic orbit in the same plane; shew that this comet cannot possibly continue within the earth's orbit longer than the $\left(\frac{2}{3\pi}\right)^{\text{th}}$ part of a year.

25. A body, projected in a given direction with a given velocity and attracted towards a given centre of force, has its velocity at every point: the velocity in a circle at the same distance $:: 1 : \sqrt{2}$; find the orbit described, the position of the apse, the magnitude of its axis, and the law of force.

26. A body acted on by a force varying partly as the inverse cube and partly as the inverse fifth power of the distance is projected with the velocity which would be acquired in falling from infinity, at an angle with the distance the

tangent of which $= \sqrt{2}$, the forces being equal at the point of projection : determine the motion.

27. A spherical surface is described in space, having in its centre a force varying as $\frac{1}{D^2}$; shew that if a particle be let fall from this surface, and be projected in any direction at any moment of its descent with the velocity acquired, it will move in an ellipse the major axis of which is equal to the radius of the sphere.

28. The force tending to the centre of a circle whose radius is c being $\mu \left(r + \frac{2c^3}{r^2} \right)$, find the velocity with which a body will describe a circle; and shew that if the velocity be suddenly doubled, it will come to an apse at the distance $3c$.

29. If a force vary inversely as the seventh power of the distance, and a body be projected from an apse with a velocity which is to the velocity in a circle at the same distance $:: 1 : \sqrt{3}$; find the equation of the curve described.

30. A body is projected from a given point with a given velocity, acted on by a central force $\propto \frac{1}{D^2}$; shew that whatever be the direction of projection, the centre of the orbit described will lie in the surface of a certain sphere.

31. If a body be projected about a centre of force varying inversely as the square of the distance, with a velocity equal to n times the velocity in a circle at the same distance, and in a direction making an angle α with the distance; the angle ϖ between the axis major and this distance may be determined from the equation

$$\tan(\varpi - \alpha) = (1 - n^2) \tan \alpha.$$

32. A body describes a parabola about a centre of force residing in a point in the circumference of a given ellipse, the foci of which are in the circumference of the parabola, the

force varying inversely as the square of the distance: shew that the time of moving from one focus to the other is the same, at whatever point in the circumference of the ellipse the centre of force is placed.

33. Two equal spheres, composed of matter attracting as $\frac{1}{D^2}$, revolve in a circle round their common centre of gravity; determine the velocity of a small body moving in the same plane in an orbit very nearly circular, at such a distance that the cube of the distance of the equal bodies divided by the distance of the third may be neglected.

34. If v be the velocity of a body revolving in an ellipse about the centre, v' its velocity when the direction of its motion is at right angles to the former direction; the time of describing the intercepted arc

$$= \frac{1}{\sqrt{\mu}} \sin^{-1} \left(\frac{vv'}{\mu ab} \right).$$

35. By supposing a comet, which really moves in a very eccentric ellipse, to move in a parabolic orbit, shew that in the computed time of passing from perihelion through 90° there will be in *ASP* an *error* of defect, *which* nearly equals $\frac{1}{10}(1 - e)$.

36. A body acted on by gravity moves on the convex surface of a cycloid, the vertex of which is its highest point; the velocity at the highest point being given; determine the point where it will leave the curve, and the latus rectum of the parabola afterwards described.

37. An ellipse is placed with its major axis in a vertical position; shew that the velocity, with which a particle must be projected vertically upwards, from the extremity of the minor axis, along the interior of the elliptic arc, so that after quitting the curve it may pass through the centre, is $\left\{ \frac{(8a^2 + b^2)g}{3a\sqrt{3}} \right\}^{\frac{1}{2}}$; a , b being the semi-axes.

38. A molecule is projected from the wheel of a coach in rapid motion; given the velocity of the coach, find the greatest distance before the wheel to which the molecule can be thrown.

39. A body moves in a groove under the action of two centres of force each varying inversely as the distance, and of equal intensity at the same distance; the body is projected from the mid-point between the centres: prove that if the velocity be uniform the form of the groove is a lemniscate.

40. A circular horizontal lamina of matter, every particle of which attracts with a force varying as $\frac{1}{\text{dist.}}$, is made to revolve with a uniform angular velocity ω round an axis through its centre perpendicular to its plane. Shew that a particle of matter will move freely along a groove (in the circle) whose equation is

$$r = a \cos \left(\frac{f}{a} \right)^{\frac{1}{2}} \frac{\theta}{\omega};$$

where f is the force at the circumference of the circle and a its radius.

41. In the last question supposing the force repulsive and a string coiled round the circle, now supposed at rest, to the extremity of which is attached a particle P , shew that when the string has been unfolded from an arc subtending an angle θ at the centre,

$$(P\text{'s velocity})^2 = f a \log_e (1 + \theta^2).$$

42. The highest point of the wheel of a carriage moves twice as fast as those points of the rim whose distances from the ground are equal to the radius.

43. A groove in the form of a cycloid with its vertex downwards and base horizontal is cut in a solid vertical board: determine the motion of a ball moving along it

while the board itself is capable of moving freely along a smooth horizontal plane; and the curve which the ball describes in space.

44. A cylinder unrolls itself from a vertical string which passes over a fixed pulley and has a weight attached to its extremity; find the tension of the string and the force accelerating its descent.

45. Two equal particles A and B , connected by a rigid line, are placed in a narrow groove, in which they can move without friction, the groove revolves with a uniform angular velocity about a vertical axis passing through a given point C in it. At the beginning of the motion $BC = 3AC$; find the time of A 's arriving at C , and the tension of the line AB at any moment.

46. A uniform lever ACB , of which the arms AC and BC are at right angles to each other, rests in equilibrium when AC is inclined at a given angle to the vertical: if AC be raised to a horizontal position (C being fixed), find the angle through which it will fall.

47. Two equal balls are attached to the extremities of a straight lever having equal arms without weight, and are each attracted to a centre of force $\propto \frac{1}{D^3}$, situated in the vertical through the fulcrum, and at a distance from it equal to the length of either arm; the rod being originally placed in a position inclined to the vertical at an angle α , shew that the time of its becoming vertical is

$$\left(\frac{a}{f}\right)^{\frac{1}{2}} \sin \alpha \cdot \log_e \tan \left(\frac{\pi}{4} + \frac{\alpha}{2}\right);$$

where a is the length of the lever and f the force of attraction on the fulcrum.

48. A uniform rod of length $2a$ slides in a vertical plane between two rough planes, one of which is vertical and the

other horizontal ; the friction on each is equal to the pressure, and the rod starts from the vertical position with a given angular velocity : determine the motion, and shew that if the given velocity be just sufficient to enable the rod to reach the horizontal position, the angular velocity at the inclination θ to the horizon will be

$$\left(\frac{6g}{37a}\right)^{\frac{1}{2}} \cdot (e^{-6\theta} + 6 \sin \theta - \cos \theta)^{\frac{1}{2}}.$$

49. A horizontal wheel moves freely about a vertical axis through its centre ; a string of definite length is wrapt round its circumference, and passing through a ring has attached to it a weight which falls by gravity ; determine the whole motion.

50. A spherical sector rests with the lowest point of its spherical surface on a smooth horizontal plane, find the time of its small oscillations.

51. A heavy chain is wound round a cylindrical axis of given weight and radius, placed in a horizontal position ; required the tension of the chain when a given length is unwound, and the velocity when the whole is run off.

52. A hemisphere rests on a horizontal plane with a string fastened to its edge, which, passing over a pulley, supports a weight : when the string is cut find the motion of the hemisphere.

53. If two bodies whose masses are m, m' , be placed on a smooth horizontal plane, and be connected by a string passing through a small ring in that plane ; and if m be projected with a velocity u in a direction perpendicular to its distance a from the ring, then when its distance is r , shew that the tension of the string $= \frac{mm'}{m+m'} \cdot \frac{(au^2)}{r^3}$. Find the time which m' takes to reach the ring, and the acquired velocity.

54. Two heavy particles are connected by a rigid rod without weight; from what point in the rod must the system be suspended that the time of its small oscillations may be a minimum?

55. Form a given quantity of matter into a cone of such dimensions that its centre of oscillation may be in its base, the point of suspension being its vertex.

56. Find the time of oscillation of a paraboloid of revolution about the directrix of the generating curve.

57. Find the moment of inertia of a cone about a slant side.

58. Find the time of the small oscillations of a uniform bent lever, the point of suspension being the angular point.

59. A heavy screw descends by its own weight; determine the motion.

60. A screw of Archimedes is capable of turning round its axis, which is vertical. A heavy particle is placed at the top of the tube and runs down through it. Determine the whole angular velocity communicated to the screw.

61. A spiral groove is made upon the surface of a heavy cone; a fine string is then wrapped upon the surface along the groove, so that the coils are close to each other, and one end of the string is made fast to the vertex, and to the other end a given weight is attached. The weight descending turns the cone about its axis, which is fixed in a horizontal position. Find the angular velocity acquired when all the string is unwrapped; and investigate the form of the path described by the descending weight.

62. A body revolving about a fixed axis, is suddenly compelled to revolve about a new axis parallel to the former: find the impulse upon the new axis at the very instant the body begins to revolve about it, and the new angular velocity.

63. A beam is drawn from a horizontal to a vertical position about one extremity, which is fixed, by means of a string which is attached to the other extremity of the beam and after passing over a pulley placed above the fixed extremity at a height equal to the length of the beam is attached to a falling body; determine the motion.

64. A cylindrical body oscillates on a plane inclined at a very small angle α to the horizon, the friction being such as to prevent sliding; its radius $= a$, and the distance of the centre of gravity from the axis $= c$; shew that the time of oscillation is very little different from that on the horizontal plane, but that the angle of vibration is *cæteris paribus* greater by $\frac{2a\alpha}{c}$.

65. A semi-cylinder rests with its plane surface on a plane, on which it is capable of moving freely; shew that a body sliding down its curved surface will describe an ellipse; and determine the time of descent.

66. A cylinder rolls down a fixed quadrant; find where the cylinder will leave the quadrant.

67. A solid cylinder rests with its base on a smooth horizontal plane: the cylinder is then hollowed so as to form a smooth hemispherical cup. A particle whose mass is equal to that of the cup, is placed at the lowest point of its internal surface; shew that if it be projected along the plane with a horizontal velocity $= 2\sqrt{ga}$ the particle will ascend just as high as the centre of the cup, and then descend.

68. A sphere revolves round an axis touching its surface, find the length of the simple isochronous pendulum.

69. A cylinder is made to rotate about its axis, and is then suddenly placed in contact with a rough horizontal plane with its axis parallel to the plane; the force of friction is of finite intensity and is not sufficiently great to prevent the line of contact of the cylinder from *sliding* on the plane at the

beginning of the motion: required to determine the motion, and to shew how long the cylinder will continue to combine a sliding motion with its rolling motion.

70. A heterogenous sphere is placed on a smooth horizontal plane in a position differing slightly from that of equilibrium. Find the time of its oscillations.

71. A cube is acted upon by three equal forces along three edges which do not meet and are not parallel to each other. Shew, that if the forces act along the three edges in the direction of the three co-ordinates x, y, z , the body revolves round an axis in the direction of the diagonal which meets none of them: but, if in the direction $x, y, -z$, the cube will move round an axis parallel to the axis of z . Find also the angular motion round the axis of rotation, the forces being supposed uniform.

72. An elastic sphere after impinging on a hard plane, not smooth, is observed to move off at an angle equal to the angle of incidence. Required the sphere's elasticity.

73. A weight P , descending vertically, draws up a weight Q , by means of a string passing over a smooth fixed peg; when Q has ascended through a given space, it suddenly becomes entangled with another weight Q' lying in its path which is drawn up with it. Find how high will Q' ascend, $Q + Q'$ being greater than P .

74. A rough vertical cylinder, capable of revolving about a concentric but smooth and smaller cylinder as an axis, rests upon a rough horizontal plane, on every point of which the pressure is the same: determine the force applied by a string wrapt round the cylinder which will just make it move. If the force be greater than this determine the motion.

75. A hemisphere oscillates about a horizontal axis, which coincides with a diameter of the base; shew that if the base be at first vertical, the ratio of the greatest pressure on the axis to the weight of the hemisphere = $109 \div 64$.

76. A cone of given form, and supported at G its centre of gravity, has a motion communicated to it round an axis through G perpendicular to the line joining G with a point in the circumference of the base, and in a plane passing through this point and the axis of the cone: determine the position of the *invariable plane*; and explain the motion of the cone's vertex.

77. Having given the values of $\Sigma(mx^2)$, $\Sigma(my^2)$, $\Sigma(mx^2)$, $\Sigma(my^2)$, $\Sigma(mxs)$, $\Sigma(mys)$, $\Sigma(mxs)$, $\Sigma(mys)$ for any system of connected particles; find the position of an axis, through the origin of co-ordinates, round which the moment of inertia of the system is a maximum or minimum.

78. A heavy body being suspended by two vertical strings, one of them suddenly breaks, compare the initial tension of the other with its tension before the accident.

79. A beam is projected in any manner along a smooth horizontal plane and impinges upon a fixed obstacle: determine the impulse.

80. At what point must a given uniform circular body be struck by a force perpendicular to its plane, that in the first instant of the body's motion one extremity of a given diameter may remain at rest?

81. A weight W is attached, by means of a string which passes over two pulleys A and B without mass and in the same horizontal line, to two equal weights P, P . If the point to which W is fixed be exactly in the middle of the horizontal portion AB of the string, and W be allowed to descend from rest, determine how far it will fall vertically.

82. If a beam lie on a smooth horizontal plane, and a perfectly elastic ball is projected against it horizontally at a given point with a given velocity and in a direction perpendicular to the beam; determine the initial motions of the beam and of the ball immediately after impact.

83. If a rough ball be projected against a rough beam on a smooth horizontal plane, determine the centre of spontaneous rotation.

84. A sphere projected along a rough plane is attracted by a force P tending to a fixed centre in that plane; prove that $p(v)^{\frac{2}{n}}$ is constant, where $n = \frac{2a^2}{a^2 + k^2}$, k being the radius of gyration and a the radius of the sphere. Shew also that the equation of the path is $\left(\frac{c}{p}\right) + n \int P = 0$.

85. A beam, moveable about a fixed horizontal axis at a given altitude above a horizontal plane, falls through a given angle: determine the point at which a given sphere should be opposed to its impact, that it may be projected to the greatest possible distance on the horizontal plane, the beam being in a vertical position at the instant of impact.

86. If a pendulum consists of two uniform metallic rods of different substances at right angles to one another, find the ratio of the lengths that a change of temperature may not effect the length of the simple pendulum; the rods being suspended at the right angle.

87. A uniform rod oscillates about one end in a vertical plane: determine the tendency to break at any point, and where this tendency is the greatest.

88. Apply the principle of least action to find the equation of the curve described by a heavy body acted on by gravity.

89. Prove, by means of the principle of least action, that the orbit of a body described about a centre of force varying inversely as the square of the distance is a conic section.

90. Find the principal axes of a right-angled triangle passing through the right angle.

91. Two equal inelastic spheres, resting on a smooth horizontal plane, are held together by means of an elastic string passing through the point of contact, one end being fastened in one sphere and the other end in the other. A third sphere moving with given velocity impinges on the system; find how it must strike the system that the string may be stretched as much as possible in consequence of the impulse.

92. Find the time in which a right cone makes small oscillations about its vertex, its axis moving in a plane. If the axis do not move in a plane, find the time.

93. If a particle placed in the plane of a triangle be attracted by the three sides, it cannot be in equilibrium unless it be in the centre of the inscribed circle.

94. A rigid uniform rod oscillates in a vertical plane about one end: determine the pressure on the axis.

95. If a beam hang vertically by means of a string attached to its upper end and to a fixed point, shew that there are two ways in which the beam can make indefinitely small oscillations so that each point of it comes to the vertical position of rest at the same instant; and find the times of oscillation.

96. A rod falling freely in a horizontal position strikes against a peg, determine the magnitude of the blow; and the subsequent motion of the rod.

97. Two wheels revolving uniformly in the same plane are suddenly brought in contact, and their axes are kept fixed, the friction being sufficient to prevent all sliding; determine what alteration is made in the angular velocities.

98. An elliptic board has the extremity of its minor axis fixed: it is struck in a direction perpendicular to its plane through one focus. If the eccentricity = $5^{-\frac{1}{2}}$ the axis of initial rotation will pass through the other focus.

99. A beam in a vertical position is moving in a horizontal direction with a given velocity, and its upper extremity becomes suddenly fixed: determine the subsequent angular velocity.

100. A rough horizontal plane is constrained to revolve round a vertical axis with a given angular velocity ω : and a uniform cylinder is placed with its base upon it. Shew that, if the friction be sufficient to prevent sliding, the cylinder will upset, when the ratio of its diameter to its height is less than $\frac{c\omega^2}{g}$, c being the distance between the axis of the cylinder and the axis of rotation.

101. If ω be variable, the cylinder will upset when the ratio of its diameter to its height is less than $\frac{c}{g} \{ \omega^4 + (d_t \omega)^2 \}^{\frac{1}{2}}$.

102. A beam, placed upon a smooth horizontal plane, has one extremity fixed, and a ball is placed in contact with it at a given distance from the fixed extremity: determine at what point another ball must impinge directly upon the beam, that the greatest possible velocity may be communicated to the first ball by the impact.

103. A body, of which the elasticity is λ , falls from rest from an altitude a in a uniform medium, the resistance of which is kv^2 ; and impinging upon a perfectly hard horizontal plane, rises and falls alternately: shew that the whole space described before the motion ceases is

$$a + \frac{1}{k} \log_e \left(\frac{1 - \lambda^2 e^{-2ka}}{1 - \lambda^2} \right).$$

104. If a heavy body, be projected horizontally with a given velocity, in a medium of which the resistance is proportional to the product of the density, into the square of the velocity; find how the density of the medium must vary that the path may be a circle, and determine the whole motion.

105. Find the attraction of a paraboloid of revolution on a particle in its focus; and the dimensions of the paraboloid that it may exert no attraction on the particle.

106. Form a given quantity of matter into a cone that shall attract a particle at its vertex with the greatest possible force.

107. A slender ring DEF is attached to another ring ABC by means of a string AD the length of which is equal to the radius of ABC ; supposing DEF to lie entirely within ABC , determine the tension of the string when DEF is at rest. Force of attraction of $ABC \propto D^{-3}$.

108. AB is a vertical line whose length is a . At the same instant that one body is suffered to fall from A another equal body is projected with a velocity u vertically upwards from B ; shew that if the resistance of the medium $= kv$, the time t in which they will meet may be found from the equation $ka = u(1 - e^{-kt})$.

109. A given quantity of matter of uniform density is to be formed into a solid of revolution: determine its figure that the attraction at one pole may be the greatest possible: the law of molecular force being that of the inverse square of the distance.

110. On a straight rod of uniform matter a small ring is placed near the middle, and then put in motion by the attraction of the rod. Determine the motion.

THE END.

